

THE STRONG MORITA EQUIVALENCE FOR COACTIONS OF A FINITE DIMENSIONAL C^* -HOPF ALGEBRA ON UNITAL C^* -ALGEBRAS

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ABSTRACT. Following Jansen and Waldmann, and Kajiwara and Watatani, we shall introduce notions of coactions of a finite dimensional C^* -Hopf algebra on a Hilbert C^* -bimodule of finite type in the sense of Kajiwara and Watatani and define their crossed product. We shall investigate their basic properties and show that the strong Morita equivalence for coactions preserves the Rohlin property for coactions of a finite dimensional C^* -Hopf algebra on unital C^* -algebras.

1. INTRODUCTION

Let A and B be unital C^* -algebras and X a Hilbert $A - B$ -bimodule of finite type in the sense of Kajiwara and Watatani [8]. Let H be a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . In this paper, following Jansen and Waldmann [7], we shall introduce the notion of coactions of H^0 on X and define their crossed product. That is, for coactions ρ and σ of H^0 on A and B , respectively, we introduce a linear map λ from X to $X \otimes H^0$, which is compatible with the coactions ρ , σ and the left A -module action, the right B -module action and the left A -valued and right B -valued inner products. Then we can define the crossed product $X \rtimes_\lambda H$, which is a Hilbert $A \rtimes_\rho H - B \rtimes_\sigma H$ -bimodule of finite type. Furthermore, we shall give a duality theorem similar to the ordinary one. This theorem in the case of countably discrete group actions and Kac systems are found in Kajiwara and Watatani [9] and Guo and Zhang [5], respectively. The latter result is almost a generalization of our duality theorem. But our approach to coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra is a useful addition to our paper, especially the main result on preservation of the Rohlin property under the strongly Morita equivalence. So, in Section 5, we give a duality theorem, a version of crossed product duality for coactions of finite dimensional C^* -Hopf algebras on Hilbert C^* -bimodules of finite type. Also, we see that if X is an $A - B$ -equivalence bimodule, we can show that

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$X \rtimes_{\lambda} H$ is an $A \rtimes_{\rho} H - B \rtimes_{\sigma} H$ -equivalence bimodule. Hence $A \rtimes_{\rho} H$ is strongly Morita equivalent to $B \rtimes_{\sigma} H$. Finally, if X is an $A - B$ -equivalence bimodule and ρ has the Rohlin property, then σ has also the Rohlin property. As an application of the result, we can obtain the following: Under a certain condition, if a unital C^* -algebra A has a finite dimensional C^* -Hopf algebra coaction of H^0 with the Rohlin property, then any unital C^* -algebra that is strongly Morita equivalent to A has also a finite dimensional C^* -Hopf algebra coaction of H^0 with the Rohlin property. In [13, Section 4], we gave an incorrect example of an action of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra with the Rohlin property. But applying the above result to [11, Section 7], we can give several examples of them.

For an algebra A , we denote by 1_A and id_A the unit element in A and the identity map on A , respectively. If no confusion arises, we denote them by 1 and id , respectively. For each $n \in \mathbf{N}$, we denote by $M_n(\mathbf{C})$ the $n \times n$ -matrix algebra over \mathbf{C} and I_n denotes the unit element in $M_n(\mathbf{C})$.

For projections p, q in a C^* -algebra A , we write $p \sim q$ in A if p and q are Murray-von Neumann equivalent in A .

2. PRELIMINARIES

Let H be a finite dimensional C^* -Hopf algebra. We denote its comultiplication, counit and antipode by Δ , ϵ and S , respectively. We shall use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for any $h \in H$ which suppresses a possible summation when we write comultiplications. We denote by N the dimension of H . Let H^0 be the dual C^* -Hopf algebra of H . We denote its comultiplication, counit and antipode by Δ^0 , ϵ^0 and S^0 , respectively. There is the distinguished projection e in H . We note that e is the Haar trace on H^0 . Also, there is the distinguished projection τ in H^0 which is the Haar trace on H . Since H is finite dimensional, $H \cong \bigoplus_{k=1}^L M_{f_k}(\mathbf{C})$ and $H^0 \cong \bigoplus_{k=1}^K M_{d_k}(\mathbf{C})$ as C^* -algebras. Let $\{v_{ij}^k \mid k = 1, 2, \dots, L, i, j = 1, 2, \dots, f_k\}$ be a system of matrix units of H . Let $\{w_{ij}^k \mid k = 1, 2, \dots, K, i, j = 1, 2, \dots, d_k\}$ be a basis of H satisfying Szymański and Peligrad's [17, Theorem 2.2,2], which is called a system of *comatrix* units of H , that is, the dual basis of a system of matrix units of H^0 . Also let $\{\phi_{ij}^k \mid k = 1, 2, \dots, K, i, j = 1, 2, \dots, d_k\}$ and $\{\omega_{ij}^k \mid k = 1, 2, \dots, L, i, j = 1, 2, \dots, f_k\}$ be systems of matrix units and comatrix units of H^0 , respectively.

Let A and B be unital C^* -algebras and X a Hilbert $A - B$ -bimodule of finite type in the sense of [8]. We regard a C^* -Hopf algebra H^0 as an $H^0 - H^0$ -equivalence bimodule in the usual way.

Let $X \otimes H^0$ be an exterior tensor product of Hilbert C^* -bimodules X and H^0 , which is a Hilbert $A \otimes H^0 - B \otimes H^0$ -bimodule.

Lemma 2.1. *With the above notations, $X \otimes H^0$ is a Hilbert $A \otimes H^0 - B \otimes H^0$ -bimodule of finite type. In particular, if X is an $A - B$ -equivalence bimodule, $X \otimes H^0$ is an $A \otimes H^0 - B \otimes H^0$ -equivalence bimodule.*

Proof. Since X is of finite type, there is a right B -basis $\{u_i\}_{i=1}^n$ of X . Then for any $x \in X$, $\phi \in H^0$,

$$\sum_{i=1}^n (u_i \otimes 1^0) \langle u_i \otimes 1^0, x \otimes \phi \rangle_{B \otimes H^0} = \sum_{i=1}^n (u_i \otimes 1^0) (\langle u_i, x \rangle_B \otimes \phi) = x \otimes \phi.$$

Thus a family $\{u_i \otimes 1^0\}_{i=1}^n$ is a right $B \otimes H^0$ -basis of $X \otimes H^0$. In the same way as above, we can see that there is a left $A \otimes H^0$ -basis of $X \otimes H^0$. Hence by [8, Proposition 1.12] or [9, Lemma 1.3], $X \otimes H^0$ is a Hilbert $A \otimes H^0 - B \otimes H^0$ -bimodule of finite type. Furthermore, we suppose that X is an $A - B$ -equivalence bimodule. Since X is full with the both-sided inner products, by the definitions of the left and right inner products of $X \otimes H^0$, so is $X \otimes H^0$. Moreover, the associativity condition of the left and right inner products of $X \otimes H^0$ holds since the associativity condition of the left and right inner products of X holds. Hence $X \otimes H^0$ is an $A \otimes H^0 - B \otimes H^0$ -equivalence bimodule. \square

Let $\text{Hom}(H, X)$ be the vector space of all linear maps from H to X . Then $X \otimes H^0$ is isomorphic to $\text{Hom}(H, X)$ as vector spaces. Sometimes, we identify $X \otimes H^0$ with $\text{Hom}(H, X)$.

3. COACTIONS OF A FINITE DIMENSIONAL C^* -HOPF ALGEBRA ON A HILBERT C^* -BIMODULE OF FINITE TYPE AND STRONG MORITA EQUIVALENCE

Let A and B be unital C^* algebras and X a Hilbert $A - B$ -bimodule of finite type. Let H be a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let ρ be a weak coaction of H^0 on A and λ a linear map from X to $X \otimes H^0$. Following [7], [9], we shall introduce the several definitions.

Definition 3.1. With the above notations, we say that $(A, X, \rho, \lambda, H^0)$ is a *weak left covariant system* if the following conditions hold:

- (1) $\lambda(ax) = \rho(a)\lambda(x)$ for any $a \in A$, $x \in X$,
- (2) $\rho_A(\langle x, y \rangle) = {}_{A \otimes H^0} \langle \lambda(x), \lambda(y) \rangle$ for any $x, y \in X$,
- (3) $(\text{id}_X \otimes \epsilon^0) \circ \lambda = \text{id}_X$.

We call λ a *weak left coaction* of H^0 on X with respect to (A, ρ) .

We define the weak action of H on A induced by ρ as follows: For any $a \in A$, $h \in H$,

$$h \cdot_\rho a = (\text{id} \otimes h)(\rho(a)),$$

where we regard H as the dual space of H^0 . In the same way as above, we can define the action of H on X induced by λ as follows: For any $x \in X$, $h \in H$,

$$h \cdot_\lambda x = (\text{id} \otimes h)(\lambda(x)) = \lambda(x)\widehat{(h)},$$

where $\lambda(x)\widehat{(\cdot)}$ is the element in $\text{Hom}(H, X)$ induced by $\lambda(x)$ in $X \otimes H^0$. Then we obtain the following conditions which are equivalent to Conditions (1)-(3) in Definition 3.1, respectively:

- (1)' $h \cdot_\lambda ax = [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\lambda x]$ for any $a \in A$, $x \in X$, $h \in H$,
- (2)' $h \cdot_\rho \langle x, y \rangle = {}_A \langle [h_{(1)} \cdot_\lambda x], [S(h_{(2)}^*) \cdot_\lambda y] \rangle$ for any $x, y \in X$, $h \in H$,
- (3)' $1_H \cdot_\lambda x = x$ for any $x \in X$.

Let σ be a weak coaction of H^0 on B .

Definition 3.2. With the above notations, we say that $(B, X, \sigma, \lambda, H^0)$ is a *weak right covariant system* if the following conditions hold:

- (4) $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B$, $x \in X$,
- (5) $\sigma(\langle x, y \rangle_B) = \langle \lambda(x), \lambda(y) \rangle_{B \otimes H^0}$ for any $x, y \in X$,
- (6) $(\text{id}_X \otimes \epsilon^0) \circ \lambda = \text{id}_X$.

We call λ a *weak right coaction* of H^0 on X with respect to (B, σ) . We can also define the weak action of H on X induced by λ satisfying the similar conditions to (1)'-(3)'. That is, we have the following conditions which are equivalent to Conditions (4)-(6), respectively:

- (4)' $h \cdot_\lambda xb = [h_{(1)} \cdot_\lambda x][h_{(2)} \cdot_\sigma b]$ for any $b \in B$, $x \in X$, $h \in H$,
- (5)' $h \cdot_\sigma \langle x, y \rangle_B = \langle [S(h_{(1)}^*) \cdot_\lambda x], [h_{(2)} \cdot_\lambda y] \rangle_B$ for any $x, y \in X$, $h \in H$,
- (6)' $1_H \cdot_\lambda x = x$ for any $x \in X$.

Let ρ and σ be weak coactions of H^0 on A and B , respectively. Let X be a Hilbert $A - B$ -bimodule of finite type.

Definition 3.3. We say that $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a *weak covariant system* if the following conditions hold:

- (1) $\lambda(ax) = \rho(a)\lambda(x)$ for any $a \in A$, $x \in X$,
- (2) $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B$, $x \in X$,
- (3) $\rho({}_A \langle x, y \rangle) = {}_{A \otimes H^0} \langle \lambda(x), \lambda(y) \rangle$ for any $x, y \in X$,
- (4) $\sigma(\langle x, y \rangle_B) = \langle \lambda(x), \lambda(y) \rangle_{B \otimes H^0}$ for any $x, y \in X$,
- (5) $(\text{id}_X \otimes \epsilon^0) \circ \lambda = \text{id}_X$.

We call λ a *weak coaction* of H^0 on X with respect to (A, B, ρ, σ) . We note that the above conditions are equivalent to the following conditions,

respectively:

- (1)' $h \cdot_\lambda ax = [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_\lambda x]$ for any $a \in A$, $x \in X$, $h \in H$,
- (2)' $h \cdot_\lambda xb = [h_{(1)} \cdot_\lambda x][h_{(2)} \cdot_\sigma b]$ for any $b \in B$, $x \in X$, $h \in H$,
- (3)' $h \cdot_\rho \langle x, y \rangle = {}_A \langle [h_{(1)} \cdot_\lambda x], [S(h_{(2)}^*) \cdot_\lambda y] \rangle$ for any $x, y \in X$, $h \in H$,
- (4)' $h \cdot_\sigma \langle x, y \rangle_B = \langle [S(h_{(1)}^*) \cdot_\lambda x], [h_{(2)} \cdot_\lambda y] \rangle_B$ for any $x, y \in X$, $h \in H$,
- (5)' $1_H \cdot_\lambda x = x$ for any $x \in X$.

We extend the above notions to coactions of a finite dimensional C^* -Hopf algebra on unital C^* -algebras.

Definition 3.4. Let A, B and H, H^0 be as above. Let ρ and σ be coactions of H^0 on A and B , respectively and let X be a Hilbert $A - B$ -bimodule of finite type.

(i) We say that $(A, X, \rho, \lambda, H^0)$ is a *left covariant system* if it is a weak left covariant system and a weak left coaction λ of H^0 on X with respect to (A, ρ) satisfies that

$$(*) \quad (\lambda \otimes \text{id}) \circ \lambda = (\text{id} \otimes \Delta^0) \circ \lambda,$$

which is equivalent to the condition that

$$(*)' \quad h \cdot_\lambda [l \cdot_\lambda x] = hl \cdot_\lambda x \text{ for any } x \in X, h, l \in H.$$

We call λ a *left coaction* of H^0 on X with respect to (A, ρ) .

(ii) We say that $(B, X, \sigma, \lambda, H^0)$ is a *right covariant system* if it is a weak right covariant system and a weak right coaction λ of H^0 on X with respect to (B, σ) satisfies $(*)$ or $(*)'$. We call λ a *right coaction* of H^0 on X with respect to (B, σ) .

(iii) We say that $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a *covariant system* if it is a weak covariant system and a weak coaction λ with respect to (A, B, ρ, σ) satisfies $(*)$ or $(*)'$. We call λ a *coaction* of H^0 on X with respect to (A, B, ρ, σ) .

Furthermore, we extend the notion of the covariant system to twisted coactions of a finite dimensional C^* -Hopf algebra on unital C^* -algebras. We recall the definition of a twisted coaction (ρ, u) of a C^* -Hopf algebra H^0 on a unital C^* -algebra A (See [9], [10]). Let ρ be a weak coaction of H^0 on A and u a unitary element in $A \otimes H^0 \otimes H^0$. Then we say that (ρ, u) is a *twisted coaction* of H^0 on A if the following conditions hold:

- (1) $(\rho \otimes \text{id}) \circ \rho = \text{Ad}(u) \circ (\text{id} \otimes \Delta^0) \circ \rho$,
- (2) $(u \otimes 1^0)(\text{id} \otimes \Delta^0 \otimes \text{id})(u) = (\rho \otimes \text{id} \otimes \text{id})(u)(\text{id} \otimes \text{id} \otimes \Delta^0)(u)$,
- (3) $(\text{id} \otimes h \otimes \epsilon^0)(u) = (\text{id} \otimes \epsilon^0 \otimes h)(u) = \epsilon^0(h)1^0$ for any $h \in H$.

The above conditions are equivalent to the following conditions, respectively:

$$(1)' \quad h \cdot_\rho [l \cdot_\rho a] = \widehat{u}(h_{(1)}, l_{(1)})[h_{(2)}l_{(2)} \cdot_\rho a]\widehat{u}^*(h_{(3)}, l_{(3)}), \text{ for any } a \in A, h, l \in H,$$

$$(2)' \quad \widehat{u}(h_{(1)}, l_{(1)})\widehat{u}(h_{(2)}, l_{(2)}, m) = [h_{(1)} \cdot_{\rho} \widehat{u}(l_{(1)}, m_{(1)})]\widehat{u}(h_{(2)}, l_{(2)}, m_{(2)}), \text{ for any } h, l, m \in H,$$

$$(3)' \quad \widehat{u}(h, 1) = \widehat{u}(1, h) = \epsilon(h)1^0 \text{ for any } h \in H.$$

Definition 3.5. Let A, B and H, H^0 be as above. Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B , respectively and let X be a Hilbert $A - B$ -bimodule of finite type. We say that $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ is a *twisted covariant system* if it is a weak covariant system and a weak coaction λ of H^0 with respect to (A, B, ρ, σ) satisfies that

$$(**) \quad (\lambda \otimes \text{id})(\lambda(x)) = u(\text{id} \otimes \Delta^0)(\lambda(x))v^* \text{ for any } x \in X,$$

which is equivalent to the condition that

$$(**)' \quad h \cdot_{\lambda} [l \cdot_{\lambda} x] = \widehat{u}(h_{(1)}, l_{(1)})[h_{(2)}, l_{(2)} \cdot_{\lambda} x]\widehat{v}^*(h_{(3)}, l_{(3)}) \text{ for any } x \in X, h, l \in H,$$

where \widehat{u} and \widehat{v} are elements in $\text{Hom}(H \times H, A)$ and $\text{Hom}(H \times H, B)$ induced by $u \in A \otimes H^0 \otimes H^0$ and $v \in B \otimes H^0 \otimes H^0$, respectively. We call λ a *twisted coaction* of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$.

Next, we introduce the notion of strong Morita equivalence for coactions of a finite dimensional C^* -Hopf algebra on unital C^* -algebras.

Definition 3.6. Let A, B and H, H^0 be as above.

(i) Let ρ and σ be weak coactions of H^0 on A and B , respectively. We say that ρ is *strongly Morita equivalent* to σ if there are an $A - B$ -equivalence bimodule X and a weak coaction λ of H^0 on X such that $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a weak covariant system.

(ii) Let ρ and σ be coactions of H^0 on A and B , respectively. We say that ρ is *strongly Morita equivalent* to σ if there are an $A - B$ -equivalence bimodule X and a coaction λ of H^0 on X such that $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a covariant system.

(iii) Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B , respectively. We say that (ρ, u) is *strongly Morita equivalent* to (σ, v) if there are an $A - B$ -equivalence bimodule X and a twisted coaction λ of H^0 on X such that $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ is a twisted covariant system.

We shall show that the above strong Morita equivalences are equivalence relations.

Proposition 3.7. *The strong Morita equivalence of weak coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra is an equivalence relation.*

Proof. It suffices to show the transitivity since the other conditions clearly hold. Let A, B, C be unital C^* -algebras and let X and Y be an $A - B$ -equivalence bimodule and a $B - C$ -equivalence bimodule, respectively. Let

ρ , σ and γ be weak coactions of H^0 on A , B and C , respectively. We suppose that ρ is strongly Morita equivalent to σ and that σ is strongly Morita equivalent to γ . Let λ and μ be weak coactions of H^0 on X and Y satisfying Definition 3.6(i), respectively. Then $X \otimes_B Y$ is an $A - C$ -equivalence bimodule. We define a bilinear map “ $\cdot_{\lambda \otimes \mu}$ ” from $H \times (X \otimes_B Y)$ to $X \otimes_B Y$ as follows: For any $x \in X$, $y \in Y$, $h \in H$,

$$h \cdot_{\lambda \otimes \mu} (x \otimes y) = [h_{(1)} \cdot_{\lambda} x] \otimes [h_{(2)} \cdot_{\mu} y].$$

Then we can show that the above map “ $\cdot_{\lambda \otimes \mu}$ ” satisfies Conditions (1)’-(5)’ in Definition 3.3 by routine computations. \square

Corollary 3.8. *The strong Morita equivalence of twisted coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra is an equivalence relation.*

Proof. By Proposition 3.7, we have only to show that Condition $(**)$ ’ in Definition 3.5. Let (ρ, u) , (σ, v) and (γ, w) be twisted coactions of H^0 on unital C^* -algebras A , B and C , respectively. Let the other notations be as in the proof of Proposition 3.7. For any $x \in X$, $y \in Y$, $h, l \in H$,

$$\begin{aligned} h \cdot_{\lambda \otimes \mu} [l \cdot_{\lambda \otimes \mu} x \otimes y] &= [h_{(1)} \cdot_{\lambda} [l_{(1)} \cdot_{\lambda} x]] \otimes [h_{(2)} \cdot_{\mu} [l_{(2)} \cdot_{\mu} y]] \\ &= \widehat{u}(h_{(1)}, l_{(1)}) [h_{(2)} l_{(2)} \cdot_{\lambda} x] \otimes [h_{(3)} l_{(3)} \cdot_{\mu} y] \widehat{w}^*(h_{(4)}, l_{(4)}) \\ &= \widehat{u}(h_{(1)}, l_{(1)}) [h_{(2)} l_{(2)} \cdot_{\lambda \otimes \mu} (x \otimes y)] \widehat{w}^*(h_{(3)}, l_{(3)}). \end{aligned}$$

Therefore, we obtain the conclusion. \square

Of course, the notion of the strong Morita equivalence of coactions of a finite dimensional C^* -Hopf algebra on unital C^* -algebras is an extension of that of actions of a finite group on unital C^* -algebras. We shall show it. Let G be a finite group and α an action of G on a unital C^* -algebra A . We consider the coaction of $C(G)$ on A induced by the action α of G on A . We denote it by the same symbol α . That is,

$$\alpha : A \longrightarrow A \otimes C(G), \quad a \longmapsto \sum_{t \in G} \alpha_t(a) \otimes \delta_t$$

for any $a \in A$, where for any $t \in G$, δ_t is a projection in $C(G)$ defined by

$$\delta_t(s) = \begin{cases} 0 & \text{if } s \neq t \\ 1 & \text{if } s = t \end{cases}.$$

Let B be a unital C^* -algebra and β an action of G on B . We denote by the same symbol β the coaction of $C(G)$ on B induced by the action β .

Proposition 3.9. *With the above notations, the following conditions are equivalent:*

- (1) *The actions α and β of G on A and B are strongly Morita equivalent,*
- (2) *The coactions α and β of $C(G)$ on A and B are strongly Morita equivalent.*

Proof. We suppose Condition (1). Then by Raeburn and Williams [15, Definition 7.2], there are an $A - B$ -equivalence bimodule X and an action u of G by linear isomorphisms of X such that

$$\alpha_t(\langle x, y \rangle) = \langle u_t(x), u_t(y) \rangle, \quad \beta_t(\langle x, y \rangle_B) = \langle u_t(x), u_t(y) \rangle_B$$

for any $x, y \in X$, $t \in G$. We note that by [15, Remark 7.3], we have the following equations:

$$u_t(ax) = \alpha_t(a)u_t(x), \quad u_t(xb) = u_t(x)\beta_t(b)$$

for any $a \in A$, $b \in B$, $x \in X$, $t \in G$. Let λ be a linear map from X to $X \otimes C(G)$ defined by for any $x \in X$,

$$\lambda(x) = \sum_{t \in G} u_t(x) \otimes \delta_t.$$

Then by routine computations, we can see that λ is a coaction of $C(G)$ on X with respect to (A, B, α, β) . Hence we obtain Condition (2). Next we suppose Condition (2). Then there are an $A - B$ -equivalence bimodule X and a coaction λ of $C(G)$ on X with respect to (A, B, α, β) . We regard G as a subset of $C^*(G)$. For any $t \in G$, we define a linear map u_t on X as follows: For any $x \in X$, $u_t(x) = t \cdot_\lambda x$. Then for any $t, s \in G$, $x \in X$,

$$u_t(u_s(x)) = t \cdot_\lambda [s \cdot_\lambda x] = ts \cdot_\lambda x = u_{ts}(x).$$

Thus we can see that u is an action of G by linear isomorphisms of X , which satisfies the desired conditions by easy computations. Thus we obtain (1). \square

Modifying [15, Example 7.4(b)], we shall obtain the following lemma, which can give examples of the strong Morita equivalence of coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra. Before it, we introduce the following definition:

Definition 3.10. Let ρ and σ be weak coactions of H^0 on A . We say that ρ is *exterior equivalent* to σ if there is a unitary element $w \in A \otimes H^0$ satisfying that

$$\sigma = \text{Ad}(w) \circ \rho, \quad (\text{id} \otimes \epsilon^0)(w) = 1.$$

Lemma 3.11. *Let ρ and σ be weak coactions of H^0 on A . Then the following conditions are equivalent:*

- (1) *The weak coactions ρ and σ are exterior equivalent,*
- (2) *The weak coactions ρ and σ are strongly Morita equivalent by a weak coaction λ from an $A - A$ -equivalence bimodule ${}_A A_A$ to an $A \otimes H^0 - A \otimes H^0$ -equivalence bimodule ${}_{A \otimes H^0} A \otimes H^0_{A \otimes H^0}$, where we regard A and $A \otimes H^0$ as an $A - A$ -equivalence bimodule and an $A \otimes H^0 - A \otimes H^0$ -equivalence bimodule in the usual way, respectively.*

Proof. We suppose Condition (1). Then there is a unitary element $w \in A \otimes H^0$ satisfying that $\sigma = \text{Ad}(w) \circ \rho$ and that $(\text{id} \otimes \epsilon^0)(w) = 1$. Let λ be a linear map from ${}_A A_A$ to ${}_{A \otimes H^0} A \otimes H^0_{A \otimes H^0}$ defined by $\lambda(x) = \rho(x)w^*$ for any $x \in {}_A A_A$. By routine computations, we can see that λ is a weak coaction of H^0 on ${}_A A_A$ with respect to (A, A, ρ, σ) . Next, we suppose Condition (2). We note that λ is a weak coaction of H^0 on ${}_A A_A$ with respect to (A, A, ρ, σ) . We identify $A \otimes H^0$ with $\text{End}_{A \otimes H^0}(A \otimes H^0_{A \otimes H^0})$, where $\text{End}_{A \otimes H^0}(A \otimes H^0_{A \otimes H^0})$ is a C^* -algebra of all right $A \otimes H^0$ -module maps on $A \otimes H^0_{A \otimes H^0}$. Let $w = \theta_{\lambda(1)^*, 1 \otimes 1^0}$ be a rank-one operator on $A \otimes H^0_{A \otimes H^0}$ induced by $\lambda(1)^*$ and $1 \otimes 1^0$. Then w is a unitary element in $\text{End}_{A \otimes H^0}(A \otimes H^0_{A \otimes H^0})$. Indeed, for any $x \in A \otimes H^0_{A \otimes H^0}$

$$\begin{aligned} ww^*(x) &= \lambda(1)^*(1 \otimes 1^0)\lambda(1)x = \langle \lambda(1), \lambda(1) \rangle_{A \otimes H^0} x = \sigma(1)x = x, \\ w^*w(x) &= \lambda(1)\lambda(1)^*x = {}_{A \otimes H^0} \langle \lambda(1), \lambda(1) \rangle x = \rho(1)x = x. \end{aligned}$$

Also, for any $a \in A$, $x \in A \otimes H^0_{A \otimes H^0}$

$$\begin{aligned} (w\rho(a)w^*)(x) &= w(\rho(a)\lambda(1)x) = \lambda(1)^*\lambda(a)x = \langle \lambda(1), \lambda(a) \rangle_{A \otimes H^0} x \\ &= \sigma(a)x. \end{aligned}$$

Thus w is a unitary element in $A \otimes H^0$ and $\sigma = \text{Ad}(w) \circ \rho$. Furthermore, let $z = (\text{id} \otimes \epsilon^0)(w)$. Then z is a unitary element in A such that $az = za$ for any $a \in A$ since $\sigma = \text{Ad}(w) \circ \rho$. Let $w_1 = w(z^* \otimes 1^0)$. Then w_1 is a unitary element in $A \otimes H^0$ satisfying that $\sigma = \text{Ad}(w_1) \circ \rho$ and that $(\text{id} \otimes \epsilon^0)(w_1) = 1$. Therefore we obtain Condition (1). \square

Lemma 3.12. *Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A . Then the following conditions are equivalent:*

- (1) *The twisted coactions (ρ, u) and (σ, v) are exterior equivalent,*
- (2) *The twisted coactions (ρ, u) and (σ, v) are strongly Morita equivalent by a twisted coaction λ from an $A - A$ -equivalence bimodule ${}_A A_A$ to an $A \otimes H^0 - A \otimes H^0$ -equivalence bimodule ${}_{A \otimes H^0} A \otimes H^0_{A \otimes H^0}$, where we regard*

A and $A \otimes H^0$ as an $A - A$ -equivalence bimodule and an $A \otimes H^0 - A \otimes H^0$ -equivalence bimodule in the usual way, respectively.

Proof. We suppose Condition (1). Then there is a unitary element $w \in A \otimes H^0$ such that

$$\sigma = \text{Ad}(w) \circ \rho, \quad v = (w \otimes 1^0)(\rho \otimes \text{id})(w)u(\text{id} \otimes \Delta^0)(w^*).$$

Let λ be as in the proof of Lemma 3.11. Then for any $x \in {}_A A_A$,

$$\begin{aligned} ((\lambda \otimes \text{id}) \circ \lambda)(x) &= u(\text{id} \otimes \Delta^0)(\rho(x))u^*(\rho \otimes \text{id})(w^*)(w^* \otimes 1^0) \\ &= u(\text{id} \otimes \Delta^0)(\rho(x))(\text{id} \otimes \Delta^0)(w^*)v^* \\ &= u(\text{id} \otimes \Delta^0)(\lambda(x))v^*. \end{aligned}$$

Thus by Lemma 3.11, λ is a twisted coaction of H^0 on ${}_A A_A$ with respect to $(A, A, \rho, u, \sigma, v)$. Next, we suppose Condition (2). We note that λ is a twisted coaction of H^0 on ${}_A A_A$ with respect to $(A, A, \rho, u, \sigma, v)$. We identify $A \otimes H^0$ with $\text{End}_{A \otimes H^0}(A \otimes H^0_{A \otimes H^0})$, where $\text{End}_{A \otimes H^0}(A \otimes H^0_{A \otimes H^0})$ is a C^* -algebra of all right $A \otimes H^0$ -module maps on $A \otimes H^0_{A \otimes H^0}$. Let $w = \theta_{\lambda(1)^*, 1 \otimes 1^0}$ be a rank-one operator on $A \otimes H^0_{A \otimes H^0}$ induced by $\lambda(1)^*$ and $1 \otimes 1^0$. Then w is a unitary element in $\text{End}_{A \otimes H^0}(A \otimes H^0_{A \otimes H^0})$ such that $\sigma = \text{Ad}(w) \circ \rho$ by Lemma 3.11. We note that $w^* = {}_{A \otimes H^0} \langle \lambda(1), 1 \otimes 1^0 \rangle$. Indeed, for any $x \in A \otimes H^0_{A \otimes H^0}$

$$w^*x = (1 \otimes 1^0) \langle \lambda(1)^*, x \rangle_{A \otimes H^0} = \lambda(1)x = {}_{A \otimes H^0} \langle \lambda(1), 1 \otimes 1^0 \rangle x.$$

Hence $w^* = {}_{A \otimes H^0} \langle \lambda(1), 1 \otimes 1^0 \rangle$. Thus

$$\begin{aligned} (\rho \otimes \text{id})(w^*) &= (\rho \otimes \text{id})({}_{A \otimes H^0} \langle \lambda(1), 1 \otimes 1^0 \rangle) \\ &= {}_{A \otimes H^0 \otimes H^0} \langle ((\lambda \otimes \text{id}) \circ \lambda)(1), \lambda(1) \otimes 1^0 \rangle \\ &= {}_{A \otimes H^0 \otimes H^0} \langle u((\text{id} \otimes \Delta^0) \circ \lambda)(1)v^*, \lambda(1) \otimes 1^0 \rangle \\ &= u((\text{id} \otimes \Delta^0) \circ \lambda)(1)v^*(\lambda(1)^* \otimes 1^0). \end{aligned}$$

It follows that

$$\begin{aligned} &(\rho \otimes \text{id})(w^*)(w^* \otimes 1^0) \\ &= u((\text{id} \otimes \Delta^0) \circ \lambda)(1)v^*(\lambda(1)^* \otimes 1^0)({}_{A \otimes H^0} \langle \lambda(1), 1 \otimes 1^0 \rangle \otimes 1^0) \\ &= u((\text{id} \otimes \Delta^0) \circ \lambda)(1)v^*(\langle \lambda(1), \lambda(1) \rangle_{A \otimes H^0} \otimes 1^0) \\ &= u((\text{id} \otimes \Delta^0) \circ \lambda)(1)v^*(\sigma(\langle 1, 1 \rangle_A) \otimes 1^0) \\ &= u(\text{id} \otimes \Delta^0)(\lambda(1))v^* = u(\text{id} \otimes \Delta^0)({}_{A \otimes H^0} \langle \lambda(1), 1 \otimes 1^0 \rangle)v^* \\ &= u(\text{id} \otimes \Delta^0)(w^*)v^*. \end{aligned}$$

Thus $v = (w \otimes 1^0)(\rho \otimes \text{id})(w)u(\text{id} \otimes \Delta^0)(w^*)$. Therefore we obtain Condition (1). \square

Next, we discuss on relations between innerness, outerness and strong Morita equivalence. Let $\rho_{H^0}^A$ be the trivial coaction of H^0 on A .

Lemma 3.13. (i) *Let ρ be a weak coaction of H^0 on A . Then the following conditions are equivalent:*

- (1) *The weak coaction ρ is inner,*
 - (2) *The weak coaction ρ is strongly Morita equivalent to $\rho_{H^0}^A$.*
- (ii) *Let ρ be a coaction of H^0 on A . Then the following conditions are equivalent:*

- (1) *The coaction ρ is strongly inner,*
- (2) *The coaction ρ is strongly Morita equivalent to $\rho_{H^0}^A$.*

Proof. (i) We suppose that ρ is inner. Then we can see that there is a unitary element $w \in A \otimes H^0$ satisfying that $\rho = \text{Ad}(w) \circ \rho_{H^0}^A$ and that $(\text{id} \otimes \epsilon^0)(w) = 1$ in the same way as in the proof that Condition (2) implies Condition (1) in Lemma 3.11. Thus ρ is exterior equivalent to $\rho_{H^0}^A$. Hence by Lemma 3.11, ρ is strongly Morita equivalent to $\rho_{H^0}^A$. Next we suppose that ρ is strongly Morita equivalent to $\rho_{H^0}^A$. Then there are an $A - A$ -equivalence bimodule X and a weak coaction λ of H^0 on X with respect to $(A, A, \rho, \rho_{H^0}^A)$. We note that for any $a \in A, x \in X$,

$$\lambda(xa) = \lambda(x)\rho_{H^0}^A(a) = \lambda(x)(a \otimes 1^0).$$

For any $h \in H$, let $\widehat{w}(h)$ be a linear map on X defined by for any $x \in X$,

$$\widehat{w}(h)x = h \cdot_\lambda x.$$

Then by the above discussion, $\widehat{w}(h) \in \text{End}_A(X)$, where $\text{End}_A(X)$ is a C^* -algebra of all right A -module maps on X . Since X is an $A - A$ -equivalence bimodule, we can identify $\text{End}_A(X)$ with A and we can regard $\widehat{w}(h)$ as an element in A for any $h \in H$. Furthermore, since the map $h \mapsto \widehat{w}(h)$ is linear, $\widehat{w} \in \text{Hom}(H, A)$. Let w be the element in $A \otimes H^0$ induced by $\widehat{w} \in \text{Hom}(H, A)$. By the definition of w , clearly $\widehat{w}(1) = 1$. We show that w is a unitary element in $A \otimes H^0$ such that $\rho = \text{Ad}(w) \circ \rho_{H^0}^A$. For any $x, y \in X, h \in H$,

$$\begin{aligned} \langle (\widehat{w^*} \widehat{w})(h)x, y \rangle_A &= \langle \widehat{w}(h_{(2)})x, \widehat{w}(S(h_{(1)}^*))y \rangle_A \\ &= \langle [h_{(2)} \cdot_\lambda x], [S(h_{(1)}^*) \cdot_\lambda y] \rangle_A = S(h^*) \cdot_{\rho_{H^0}^A} \langle x, y \rangle_A = \langle \epsilon(h)x, y \rangle_A. \end{aligned}$$

Thus $w^*w = 1 \otimes 1^0$. Also, for any $x, y \in X, h \in H$,

$$\begin{aligned} h \cdot_\rho \langle x, y \rangle &= {}_A \langle [h_{(1)} \cdot_\lambda x], [S(h_{(2)}^*) \cdot_\lambda y] \rangle = {}_A \langle \widehat{w}(h_{(1)})x, \widehat{w}(S(h_{(2)}^*))y \rangle \\ &= \widehat{w}(h_{(1)})_A \langle x, y \rangle \widehat{w^*}(h_{(2)}). \end{aligned}$$

Hence $\rho = \text{Ad}(w) \circ \rho_{H^0}^A$ since X is an $A - A$ -equivalence bimodule. Thus $ww^* = w\rho_{H^0}^A(1)w^* = \rho(1) = 1 \otimes 1^0$. Therefore, the weak coaction ρ is inner.

(ii) We suppose that a coaction ρ is strongly inner. Then ρ is exterior equivalent to $\rho_{H^0}^A$. Hence by Lemma 3.12, ρ is strongly Morita equivalent to $\rho_{H^0}^A$. Next, we suppose that ρ is strongly Morita equivalent to $\rho_{H^0}^A$. Then there are an $A - A$ -equivalence bimodule X and a coaction λ of H^0 on X with respect to $(A, A, \rho, \rho_{H^0}^A)$. Let w be as in (i). It suffices to show that for any $h, l \in H$, $\widehat{w}(hl) = \widehat{w}(h)\widehat{w}(l)$. For any $x \in X$, $h, l \in H$,

$$\widehat{w}(h)\widehat{w}(l)x = h \cdot_\lambda [l \cdot_\lambda x] = hl \cdot_\lambda x = \widehat{w}(hl)x.$$

Therefore, ρ is strongly inner. \square

Let $\rho_{H^0}^A$ and $\rho_{H^0}^B$ be the trivial coactions of H^0 on A and B , respectively. We suppose that A and B are strongly Morita equivalent and let X be an $A - B$ -equivalence bimodule. Then $\rho_{H^0}^A$ and $\rho_{H^0}^B$ are strongly Morita equivalent. If a linear map $\lambda_{H^0}^X$ from X to $X \otimes H^0$ is defined by $\lambda_{H^0}^X(x) = x \otimes 1^0$ for any $x \in X$, then the $\lambda_{H^0}^X$ is a coaction of H^0 on X with respect to $(A, B, \rho_{H^0}^A, \rho_{H^0}^B)$.

Corollary 3.14. (i) *Let ρ and σ be weak coactions of H^0 on A and B , respectively. If ρ is strongly Morita equivalent to σ , then ρ is inner if and only if so is σ .*

(ii) *Let ρ and σ be coactions of H^0 on A and B , respectively. If ρ is strongly Morita equivalent to σ , then ρ is strongly inner if and only if so is σ .*

Proof. (i) We suppose that ρ is inner. Then σ is strongly Morita equivalent to $\rho_{H^0}^B$ by Lemma 3.13(i), Proposition 3.7 and the above discussion. Therefore, σ is inner by Lemma 3.13(i).

(ii) We suppose that ρ is strongly inner. Then σ is strongly Morita equivalent to $\rho_{H^0}^B$ by Lemma 3.13(ii), Corollary 3.8 and the above discussion. Therefore, σ is strongly inner by Lemma 3.13(ii). \square

Proposition 3.15. *We suppose that H^0 is not trivial. Let ρ and σ be coactions of H^0 on A and B , respectively. If ρ is strongly Morita equivalent to σ , then ρ is outer if and only if so is σ .*

Proof. We suppose that ρ is outer. We show that σ is outer. Let π be a surjective C^* -Hopf algebra homomorphism of H^0 onto a non-trivial C^* -Hopf algebra K^0 . We suppose that $(\text{id} \otimes \pi) \circ \sigma$ is inner. Then $(\text{id} \otimes \pi) \circ \sigma$ is strongly Morita equivalent to $(\text{id} \otimes \pi) \circ \rho$ by easy computations. Thus by Corollary 3.14(i), $(\text{id} \otimes \pi) \circ \rho$ is inner. This is a contradiction. Therefore, we obtain the conclusion. \square

Furthermore, we have also the following easy lemma:

Lemma 3.16. *Let (ρ, u) be a twisted coaction of H^0 on A and let $(\rho \otimes \text{id}, u \otimes I_n)$ be a twisted coaction of H^0 on $A \otimes M_n(\mathbf{C})$, where n is any positive integer and we identify $A \otimes H^0 \otimes M_n(\mathbf{C})$ with $A \otimes M_n(\mathbf{C}) \otimes H^0$. Then (ρ, u) is strongly Morita equivalent to $(\rho \otimes \text{id}, u \otimes I_n)$.*

Proof. Let f be a minimal projection in $M_n(\mathbf{C})$ and let $X = (1 \otimes f)(A \otimes M_n(\mathbf{C}))$. We regard it as an $A - A \otimes M_n(\mathbf{C})$ -equivalence bimodule in the usual way. Let λ be a linear map from X to $X \otimes H^0$ defined by

$$\lambda((1 \otimes f)x) = (1 \otimes f \otimes 1^0)(\rho \otimes \text{id})(x)$$

for any $x \in A \otimes M_n(\mathbf{C})$, where we identify $A \otimes H^0 \otimes M_n(\mathbf{C})$ with $A \otimes M_n(\mathbf{C}) \otimes H^0$. By routine computations, we can see that λ satisfies Conditions (1)-(5) in Definition 3.3 and Condition (**). \square

4. CROSSED PRODUCTS OF HILBERT C^* -BIMODULES OF FINITE TYPE BY FINITE DIMENSIONAL C^* -HOPF ALGEBRAS

In this section, we extend the notion of crossed products of Hilbert C^* -bimodules of finite type defined in [7], [9] to (twisted) coactions of finite dimensional C^* -Hopf algebras.

Let H be a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let A and B be unital C^* -algebras and X a Hilbert $A - B$ -bimodule of finite type. Let $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ be a twisted covariant system. Under certain conditions, we define $X \rtimes_\lambda H$, a Hilbert $A \rtimes_{\rho, u} H - B \rtimes_{\sigma, v} H$ -bimodule of finite type as follows: $X \rtimes_\lambda H$ is just $X \otimes H$ (the algebraic tensor product) as vector spaces. Its left action and right action are given by

$$\begin{aligned} (a \rtimes_{\rho, u} h)(x \rtimes_\lambda l) &= a[h_{(1)} \cdot_\lambda x] \widehat{v}(h_{(2)}, l_{(1)}) \rtimes_\lambda h_{(3)} l_{(2)}, \\ (x \rtimes_\lambda l)(b \rtimes_{\sigma, v} m) &= x[l_{(1)} \cdot_{\sigma, v} b] \widehat{v}(l_{(2)}, m_{(1)}) \rtimes_\lambda l_{(3)} m_{(2)} \end{aligned}$$

for any $a \in A$, $b \in B$, $x \in X$ and $h, l, m \in H$. Then for any $a_1, a_2 \in A$, $x \in X$, $h, l, m \in H$,

$$\begin{aligned} &((a_1 \rtimes_{\rho, u} h)(a_2 \rtimes_{\rho, u} l))(x \rtimes_\lambda m) \\ &= a_1[h_{(1)} \cdot_{\rho, u} a_2] \widehat{u}(h_{(2)}, l_{(1)})[h_{(3)} l_{(2)} \cdot_\lambda x] \widehat{v}(h_{(4)} l_{(3)}, m_{(1)}) \rtimes_\lambda h_{(5)} l_{(4)} m_{(2)} \\ &= a_1[h_{(1)} \cdot_\lambda a_2[l_{(1)} \cdot_\lambda x]] \widehat{v}(h_{(2)}, l_{(2)}) \widehat{v}(h_{(3)} l_{(3)}, m_{(1)}) \rtimes_\lambda h_{(4)} l_{(4)} m_{(2)} \\ &= a_1[h_{(1)} \cdot_\lambda a_2[l_{(1)} \cdot_\lambda x] \widehat{v}(l_{(2)}, m_{(1)})] \widehat{v}(h_{(2)}, l_{(3)} m_{(2)}) \rtimes_\lambda h_{(3)} l_{(4)} m_{(3)} \\ &= (a_1 \rtimes_{\rho, u} h)((a_2 \rtimes_{\rho, u} l)(x \rtimes_\lambda m)). \end{aligned}$$

Also, for any $b_1, b_2 \in B$, $x \in X$, $h, l, m \in H$,

$$\begin{aligned}
& (x \rtimes_\lambda h)((b_1 \rtimes_{\sigma,v} l)(b_2 \rtimes_{\sigma,v} m)) \\
&= x[h_{(1)} \cdot_{\sigma,v} b_1][h_{(2)} \cdot_{\sigma,v} [l_{(1)} \cdot_{\sigma,v} b_2]]\widehat{v}(h_{(3)}, l_{(2)})\widehat{v}(h_{(4)}l_{(3)}, m_{(1)}) \\
&\quad \rtimes_\lambda h_{(5)}l_{(4)}m_{(2)} \\
&= x[h_{(1)} \cdot_{\sigma,v} b_1]\widehat{v}(h_{(2)}, l_{(1)})[h_{(3)}l_{(2)} \cdot_{\sigma,v} b_2]\widehat{v}(h_{(4)}l_{(3)}, m_{(1)}) \rtimes_\lambda h_{(5)}l_{(4)}m_{(2)} \\
&= ((x \rtimes_\lambda h)(b_1 \rtimes_{\sigma,v} l))(b_2 \rtimes_{\sigma,v} m).
\end{aligned}$$

Thus $X \rtimes_\lambda H$ is a left $A \rtimes_{\rho,u} H$ and right $B \rtimes_{\sigma,v} H$ -bimodule. Also, its left $A \rtimes_{\rho,u} H$ -valued inner product and right $B \rtimes_{\sigma,v} H$ -valued inner product are given by

$$\begin{aligned}
& {}_{A \rtimes_{\rho,u} H} \langle x \rtimes_\lambda h, y \rtimes_\lambda l \rangle \\
&= {}_A \langle x, [S(h_{(2)}l_{(3)}^*)^* \cdot_\lambda y]\widehat{v}(S(h_{(1)}l_{(2)}^*)^*, l_{(1)}) \rangle \rtimes_{\rho,u} h_{(3)}l_{(4)}^*, \\
&\langle x \rtimes_\lambda h, y \rtimes_\lambda l \rangle_{B \rtimes_{\sigma,v} H} \\
&= \widehat{v}^*(h_{(2)}^*, S(h_{(1)}^*)) [h_{(3)}^* \cdot_{\sigma,v} \langle x, y \rangle_B] \widehat{v}(h_{(4)}^*, l_{(1)}) \rtimes_{\sigma,v} h_{(5)}^*l_{(2)}
\end{aligned}$$

for any $x, y \in X$ and $h, l \in H$. We shall show that $X \rtimes_\lambda H$ is a Hilbert $A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H$ -bimodule of finite type proving that $X \rtimes_\lambda H$ satisfies Conditions (1)-(10) in [9, Lemma 1.3]. Clearly $X \rtimes_\lambda H$ is a left $A \rtimes_\rho H$ - and right $B \rtimes_\sigma H$ -bimodule. Thus Conditions (1), (4) in [9, Lemma 1.3] are satisfied. For any $a, b \in A$, $x, y \in X$ and $h, l, m \in H$,

$$\begin{aligned}
& (a \rtimes_{\rho,u} h) {}_{A \rtimes_{\rho,u} H} \langle x \rtimes_\lambda l, y \rtimes_\lambda m \rangle \\
&= a[h_{(1)} \cdot_{\rho,u} {}_A \langle x, [S(l_{(2)}m_{(3)}^*)^* \cdot_\lambda y]\widehat{v}(S(l_{(1)}m_{(2)}^*)^*, m_{(1)}) \rangle] \widehat{u}(h_{(2)}, l_{(3)}m_{(4)}^*) \\
&\quad \rtimes_{\rho,u} h_{(3)}l_{(4)}m_{(5)}^* \\
&= a {}_A \langle [h_{(1)} \cdot_\lambda x], [S(h_{(3)}^*) \cdot_\lambda [S(l_{(2)}m_{(3)}^*)^* \cdot_\lambda y][S(h_{(2)}^*) \cdot_{\sigma,v} \widehat{v}(S(l_{(1)}m_{(2)}^*)^*, m_{(1)})]] \rangle \\
&\quad \times \widehat{u}(h_{(4)}, l_{(3)}m_{(4)}^*) \rtimes_{\rho,u} h_{(5)}l_{(4)}m_{(5)}^* \\
&= a {}_A \langle [h_{(1)} \cdot_\lambda x], [S(h_{(5)}^*) \cdot_\lambda [S(l_{(4)}m_{(4)}^*)^* \cdot_\lambda y]]\widehat{v}(S(h_{(4)}^*), S(l_{(3)}m_{(3)}^*)^*) \rangle \\
&\quad \times \widehat{v}(S(h_{(3)}l_{(2)}m_{(2)}^*)^*, m_{(1)})\widehat{v}^*(S(h_{(2)}^*), S(l_{(1)}^*)) \rangle \widehat{u}(h_{(6)}, l_{(5)}m_{(5)}^*) \\
&\quad \rtimes_{\rho,u} h_{(7)}l_{(6)}m_{(6)}^* \\
&= a {}_A \langle [h_{(1)} \cdot_\lambda x], \widehat{u}(S(h_{(5)}^*), S(l_{(4)}m_{(4)}^*)^*)[S(h_{(4)}l_{(3)}m_{(3)}^*)^* \cdot_\lambda y] \\
&\quad \times \widehat{v}(S(h_{(3)}l_{(2)}m_{(2)}^*)^*, m_{(1)})\widehat{v}(h_{(2)}, l_{(1)})^* \rangle \widehat{u}(h_{(6)}, l_{(5)}m_{(5)}^*) \rtimes_{\rho,u} h_{(7)}l_{(6)}m_{(6)}^* \\
&= a {}_A \langle [h_{(1)} \cdot_\lambda x]\widehat{v}(h_{(2)}, l_{(1)}), [S(h_{(4)}l_{(3)}m_{(3)}^*)^* \cdot_\lambda y]\widehat{v}(S(h_{(3)}l_{(2)}m_{(2)}^*)^*, m_{(1)}) \rangle \\
&\quad \times \widehat{u}^*(h_{(5)}, l_{(4)}m_{(4)}^*)\widehat{u}(h_{(6)}, l_{(5)}m_{(5)}^*) \rtimes_{\rho,u} h_{(7)}l_{(6)}m_{(6)}^* \\
&= a {}_A \langle [h_{(1)} \cdot_\lambda x]\widehat{v}(h_{(2)}, l_{(1)}), [S(h_{(4)}l_{(3)}m_{(3)}^*)^* \cdot_\lambda y]\widehat{v}(S(h_{(3)}l_{(2)}m_{(2)}^*)^*, m_{(1)}) \rangle \\
&\quad \rtimes_{\rho,u} h_{(5)}l_{(4)}m_{(4)}^* \\
&= {}_{A \rtimes_{\rho,u} H} \langle a[h_{(1)} \cdot_\lambda x]\widehat{v}(h_{(2)}, l_{(1)}) \rtimes_\lambda h_{(3)}l_{(2)}, y \rtimes_\lambda m \rangle
\end{aligned}$$

$$= A \rtimes_{\rho, u} H \langle (a \rtimes_{\rho, u} h)(x \rtimes_{\lambda} l), y \rtimes_{\lambda} m \rangle.$$

Also,

$$\begin{aligned} & \langle x \rtimes_{\lambda} h, y \rtimes_{\lambda} l \rangle_{B \rtimes_{\sigma, v} H} (b \rtimes_{\sigma, v} m) \\ &= \widehat{v}^*(h_{(2)}^*, S(h_{(1)}^*)) [h_{(3)}^* \cdot_{\sigma, v} \langle x, y \rangle_B] \widehat{v}(h_{(4)}^*, l_{(1)}) [h_{(5)}^* l_{(2)} \cdot_{\sigma, v} b] \widehat{v}(h_{(6)}^* l_{(3)}, m_{(1)}) \\ & \rtimes_{\sigma, v} h_{(7)}^* l_{(4)} m_{(2)} \\ &= \widehat{v}^*(h_{(2)}^*, S(h_{(1)}^*)) [h_{(3)}^* \cdot_{\sigma, v} \langle x, y \rangle_B] [h_{(4)}^* \cdot_{\sigma, v} [l_{(1)} \cdot_{\sigma, v} b]] \widehat{v}(h_{(5)}^*, l_{(2)}) \widehat{v}(h_{(6)}^* l_{(3)}, m_{(1)}) \\ & \rtimes_{\sigma, v} h_{(7)}^* l_{(4)} m_{(2)} \\ &= \widehat{v}^*(h_{(2)}^*, S(h_{(1)}^*)) [h_{(3)}^* \cdot_{\sigma, v} \langle x, y \rangle_B] [l_{(1)} \cdot_{\sigma, v} b]] \widehat{v}(h_{(4)}^*, l_{(2)}) \widehat{v}(h_{(5)}^* l_{(3)}, m_{(1)}) \\ & \rtimes_{\sigma, v} h_{(6)}^* l_{(4)} m_{(2)} \\ &= \widehat{v}^*(h_{(2)}^*, S(h_{(1)}^*)) [h_{(3)}^* \cdot_{\sigma, v} \langle x, y \rangle_B] [l_{(1)} \cdot_{\sigma, v} b]] [h_{(4)}^* \cdot_{\sigma, v} \widehat{v}(l_{(2)}, m_{(1)})] \widehat{v}(h_{(5)}^*, l_{(3)} m_{(2)}) \\ & \rtimes_{\sigma, v} h_{(6)}^* l_{(4)} m_{(3)} \\ &= \widehat{v}^*(h_{(2)}^*, S(h_{(1)}^*)) [h_{(3)}^* \cdot_{\sigma, v} \langle x, y \rangle_B] [l_{(1)} \cdot_{\sigma, v} b]] \widehat{v}(l_{(2)}, m_{(1)}) \widehat{v}(h_{(4)}^*, l_{(3)} m_{(2)}) \\ & \rtimes_{\sigma, v} h_{(5)}^* l_{(4)} m_{(3)} \\ &= \langle x \rtimes_{\lambda} h, (y \rtimes_{\lambda} l)(b \rtimes_{\sigma, v} m) \rangle_{B \rtimes_{\sigma, v} H}. \end{aligned}$$

Thus Conditions (3), (6) in [9, Lemma 1.3] are satisfied. For any $x, y \in X$ and $h, l \in H$,

$$\begin{aligned} & A \rtimes_{\rho, u} H \langle x \rtimes_{\lambda} h, y \rtimes_{\lambda} l \rangle^* \\ &= \widehat{u}^*(l_{(5)} h_{(4)}^*, S(l_{(4)} h_{(3)}^*)) [l_{(6)} h_{(5)}^* \cdot_{\rho, u} A \langle [S(l_{(3)} h_{(2)}^*) \cdot_{\lambda} y] \widehat{v}(S(l_{(2)} h_{(1)}^*), l_{(1)}), x \rangle] \\ & \rtimes_{\rho, u} l_{(7)} h_{(6)}^* \\ &= \widehat{u}^*(l_{(5)} h_{(4)}^*, S(l_{(4)} h_{(3)}^*)) \\ & \times A \langle [l_{(6)} h_{(5)}^* \cdot_{\lambda} [S(l_{(3)} h_{(2)}^*) \cdot_{\lambda} y]] [l_{(7)} h_{(6)}^* \cdot_{\sigma, v} \widehat{v}(S(l_{(2)} h_{(1)}^*), l_{(1)})], [S(l_{(8)} h_{(7)}^*)^* \cdot_{\lambda} x] \rangle \\ & \rtimes_{\rho, u} l_{(9)} h_{(8)}^* \\ &= \widehat{u}^*(l_{(6)} h_{(6)}^*, S(l_{(5)} h_{(5)}^*)) \\ & \times A \langle [l_{(7)} h_{(7)}^* \cdot_{\lambda} [S(l_{(4)} h_{(4)}^*) \cdot_{\lambda} y]] \widehat{v}(l_{(8)} h_{(8)}^*, S(l_{(3)} h_{(3)}^*)) \widehat{v}(l_{(9)} h_{(9)}^* S(l_{(2)} h_{(2)}^*), l_{(1)}) \\ & \times \widehat{v}^*(l_{(10)} h_{(10)}^*, S(h_{(1)}^*)), [S(l_{(11)} h_{(11)}^*)^* \cdot_{\lambda} x] \rangle \rtimes_{\rho, u} l_{(12)} h_{(12)}^* \\ &= \widehat{u}^*(l_{(6)} h_{(6)}^*, S(l_{(5)} h_{(5)}^*)) \\ & \times A \langle \widehat{u}(l_{(7)} h_{(7)}^*, S(l_{(4)} h_{(4)}^*)) [l_{(8)} h_{(8)}^* S(l_{(3)} h_{(3)}^*) \cdot_{\lambda} y] \widehat{v}(l_{(9)} h_{(9)}^* S(l_{(2)} h_{(2)}^*), l_{(1)}) \\ & \times \widehat{v}^*(l_{(10)} h_{(10)}^*, S(h_{(1)}^*)), [S(l_{(11)} h_{(11)}^*)^* \cdot_{\lambda} x] \rangle \rtimes_{\rho, u} l_{(8)} h_{(8)}^* \\ &= A \langle y, [S(l_{(2)} h_{(3)}^*)^* \cdot_{\lambda} x] \widehat{v}(S(l_{(1)} h_{(2)}^*)^*, h_{(1)}) \rangle \rtimes_{\rho, u} l_{(3)} h_{(4)}^* \\ &= A \rtimes_{\rho, u} H \langle y \rtimes_{\lambda} l, x \rtimes_{\lambda} h \rangle. \end{aligned}$$

Similarly

$$\begin{aligned}
& \langle x \rtimes_\lambda h, y \rtimes_\lambda l \rangle_{B \rtimes_{\sigma, v} H}^* \\
&= \widehat{v}^*(l_{(3)}^* h_{(6)}, S(l_{(2)}^* h_{(5)})) [l_{(4)}^* h_{(7)} \cdot_\sigma \widehat{v}(h_{(4)}^*, l_{(1)})^* [S(h_{(3)}) \cdot_{\sigma, v} \langle y, x \rangle_B] \widehat{v}(S(h_{(2)}), h_{(1)})] \\
&\quad \rtimes_{\sigma, v} l_{(5)}^* h_{(8)} \\
&= \widehat{v}^*(l_{(3)}^* h_{(6)}, S(l_{(2)} h_{(5)})) [l_{(4)}^* h_{(7)} \cdot_{\sigma, v} \widehat{v}^*(S(h_{(4)}), S(l_{(1)}^*))] \\
&\quad \times [l_{(5)}^* h_{(8)} \cdot_{\sigma, v} [S(h_{(3)}) \cdot_{\sigma, v} \langle y, x \rangle_B]] [l_{(6)}^* h_{(9)} \cdot_{\sigma, v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma, v} l_{(7)}^* h_{(10)} \\
&= \widehat{v}(S(l_{(3)}^* h_{(6)})^*, (l_{(2)}^* h_{(5)})^*)^* [S(h_{(7)}^* l_{(4)}) \cdot_\sigma \widehat{v}(h_{(4)}^*, l_{(1)})]^* \\
&\quad \times [l_{(5)}^* h_{(8)} \cdot_{\sigma, v} [S(h_{(3)}) \cdot_{\sigma, v} \langle y, x \rangle_B]] [l_{(6)}^* h_{(9)} \cdot_{\sigma, v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma, v} l_{(7)}^* h_{(10)} \\
&= [[S(h_{(7)}^* l_{(4)}) \cdot_{\sigma, v} \widehat{v}(h_{(4)}^*, l_{(1)})] \widehat{v}(S(h_{(6)}^* l_{(3)}), h_{(5)}^* l_{(2)})]^* [l_{(5)}^* h_{(8)} \cdot_{\sigma, v} [S(h_{(3)}) \cdot_{\sigma, v} \langle y, x \rangle_B]] \\
&\quad \times [l_{(6)}^* h_{(9)} \cdot_{\sigma, v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma, v} l_{(7)}^* h_{(10)} \\
&= [\widehat{v}(S(h_{(7)}^* l_{(3)}), h_{(4)}^*) \widehat{v}(S(h_{(6)}^* l_{(2)}) h_{(5)}^*, l_{(1)})]^* [l_{(4)}^* h_{(8)} \cdot_{\sigma, v} [S(h_{(3)}) \cdot_{\sigma, v} \langle y, x \rangle_B]] \\
&\quad \times [l_{(5)}^* h_{(9)} \cdot_{\sigma, v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma, v} l_{(6)}^* h_{(10)} \\
&= \widehat{v}(S(l_{(2)}), l_{(1)})^* \widehat{v}^*(l_{(3)}^* h_{(5)}, S(h_{(4)})) [l_{(4)}^* h_{(6)} \cdot_{\sigma, v} [S(h_{(3)}) \cdot_{\sigma, v} \langle y, x \rangle_B]] \\
&\quad \times [l_{(5)}^* h_{(7)} \cdot_{\sigma, v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma, v} l_{(6)}^* h_{(8)} \\
&= \widehat{v}(S(l_{(2)}), l_{(1)})^* [l_{(3)}^* h_{(5)} S(h_{(4)}) \cdot_{\sigma, v} \langle y, x \rangle_B] \widehat{v}^*(l_{(4)}^* h_{(6)}, S(h_{(3)})) \\
&\quad \times [l_{(5)}^* h_{(7)} \cdot_{\sigma, v} \widehat{v}(S(h_{(2)}), h_{(1)})] \rtimes_{\sigma, v} l_{(6)}^* h_{(8)} \\
&= \widehat{v}(S(l_{(2)}), l_{(1)})^* [l_{(3)}^* \cdot_{\sigma, v} \langle y, x \rangle_B] \widehat{v}^*(l_{(4)}^* h_{(4)}, S(h_{(3)})) [l_{(5)}^* h_{(5)} \cdot_{\sigma, v} \widehat{v}(S(h_{(2)}), h_{(1)})] \\
&\quad \rtimes_{\sigma, v} l_{(6)}^* h_{(6)} \\
&= \widehat{v}(S(l_{(2)}), l_{(1)})^* [l_{(3)}^* \cdot_{\sigma, v} \langle y, x \rangle_B] \widehat{v}(l_{(4)}^* h_{(5)} S(h_{(4)}), h_{(1)}) \widehat{v}^*(l_{(5)}^* h_{(6)}, S(h_{(3)}) h_{(2)}) \\
&\quad \rtimes_{\sigma, v} l_{(6)}^* h_{(7)} \\
&= \langle y \rtimes_\lambda l, x \rtimes_\lambda h \rangle_{B \rtimes_{\sigma, v} H}.
\end{aligned}$$

Thus Conditions (2), (5) in [9, Lemma 1.3] are satisfied. Moreover, for any $b \in B, x, y \in X, l, m \in H$,

$$\begin{aligned}
& {}_{A \rtimes_{\rho, u} H} \langle x \rtimes_\lambda l, (y \rtimes_\lambda m)(b \rtimes_{\sigma, v} 1)^* \rangle \\
&= {}_A \langle x, [S(l_{(3)} m_{(5)}^*)^* \cdot_\lambda y] \widehat{v}(S(l_{(2)} m_{(4)}^*)^*, m_{(1)}) [S(l_{(1)} m_{(3)}^*)^* m_{(2)} \cdot_{\sigma, v} b^*] \rangle \\
&\quad \rtimes_{\rho, u} l_{(4)} m_{(6)}^* \\
&= {}_A \langle x [l_{(1)} \cdot_{\sigma, v} b], [S(l_{(3)} m_{(3)}^*)^* \cdot_\lambda y] \widehat{v}(S(l_{(2)} m_{(2)}^*)^*, m_{(1)}) \rangle \rtimes_{\rho, u} l_{(4)} m_{(4)}^* \\
&= {}_{A \rtimes_{\rho, u} H} \langle (x \rtimes_\lambda l)(b \rtimes_{\sigma, v} 1), y \rtimes_\lambda m \rangle.
\end{aligned}$$

Also, for any $x, y \in X$, $h, l, m \in H$,

$$\begin{aligned}
& A \rtimes_{\rho, u} H \langle x \rtimes_{\lambda} l, (y \rtimes_{\lambda} m)(1 \rtimes_{\sigma, v} h)^* \rangle \\
&= A \rtimes_{\rho, u} H \langle x \rtimes_{\lambda} l, y[m_{(1)} \cdot_{\sigma, v} \widehat{v}(S(h_{(2)}), h_{(1)})^*] \widehat{v}(m_{(2)}, h_{(3)}^*) \rtimes_{\lambda} m_{(3)} h_{(4)}^* \rangle \\
&= A \langle x, [S(l_{(2)} h_{(6)} m_{(5)}^*)^* \cdot_{\lambda} y[m_{(1)} \cdot_{\sigma, v} \widehat{v}^*(h_{(2)}^*, S(h_{(1)}^*))] \widehat{v}(m_{(2)}, h_{(3)}^*)] \\
&\quad \times \widehat{v}(S(l_{(1)} h_{(5)} m_{(4)}^*)^*, m_{(3)} h_{(4)}^*) \rtimes_{\rho, u} l_{(3)} h_{(7)} m_{(6)}^* \rangle \\
&= A \langle x, [S(l_{(2)} h_{(7)} m_{(5)}^*)^* \cdot_{\lambda} y \widehat{v}(m_{(1)}, h_{(3)}^* S(h_{(2)}^*)) \widehat{v}^*(m_{(2)} h_{(4)}^*, S(h_{(1)}^*))] \\
&\quad \times \widehat{v}(S(l_{(1)} h_{(6)} m_{(4)}^*)^*, m_{(3)} h_{(5)}^*) \rtimes_{\rho, u} l_{(3)} h_{(8)} m_{(6)}^* \rangle \\
&= A \langle x, [S(l_{(3)} h_{(6)} m_{(5)}^*)^* \cdot_{\lambda} y][S(l_{(2)} h_{(5)} m_{(4)}^*)^* \cdot_{\sigma, v} \widehat{v}^*(m_{(1)} h_{(2)}^*, S(h_{(1)}^*))] \\
&\quad \times \widehat{v}(S(l_{(1)} h_{(4)} m_{(3)}^*)^*, m_{(2)} h_{(3)}^*) \rtimes_{\rho, u} l_{(4)} h_{(7)} m_{(6)}^* \rangle \\
&= A \langle x, [S(l_{(3)} h_{(7)} m_{(5)}^*)^* \cdot_{\lambda} y] \widehat{v}(S(l_{(2)} h_{(6)} m_{(4)}^*)^*, m_{(1)} h_{(3)}^* S(h_{(2)}^*)) \\
&\quad \times \widehat{v}^*(S(l_{(1)} h_{(5)} m_{(3)}^*)^* m_{(2)} h_{(4)}^*, S(h_{(1)}^*)) \rtimes_{\rho, u} l_{(4)} h_{(8)} m_{(6)}^* \rangle \\
&= A \langle x \widehat{v}(l_{(1)}, h_{(1)}), [S(l_{(3)} h_{(3)} m_{(3)}^*)^* \cdot_{\lambda} y] \widehat{v}(S(l_{(2)} h_{(2)} m_{(2)}^*)^*, m_{(1)}) \rangle \\
&\quad \rtimes_{\rho, u} l_{(4)} h_{(4)} m_{(4)}^* \rangle \\
&= A \rtimes_{\rho, u} H \langle (x \rtimes_{\lambda} l)(1 \rtimes_{\sigma, v} h), y \rtimes_{\lambda} m \rangle.
\end{aligned}$$

Thus we obtain that for any $b \in B$, $x, y \in X$, $h, l, m \in H$,

$$A \rtimes_{\rho, u} H \langle (x \rtimes_{\lambda} l)(b \rtimes_{\sigma, v} h), y \rtimes_{\lambda} m \rangle = A \rtimes_{\rho, u} H \langle x \rtimes_{\lambda} l, (y \rtimes_{\lambda} m)(b \rtimes_{\sigma, v} h)^* \rangle.$$

We note that for any $a \in A$, $x, y \in X$, $h, l, m \in H$,

$$\begin{aligned}
& \langle (a \rtimes_{\rho, u} h)(x \rtimes_{\lambda} l), y \rtimes_{\lambda} m \rangle_{B \rtimes_{\sigma, v} H} \\
&= (1 \rtimes_{\sigma, v} l)^* \langle (a \rtimes_{\rho, u} h)(x \rtimes_{\lambda} 1), y \rtimes_{\lambda} 1 \rangle_{B \rtimes_{\sigma, v} H} (1 \rtimes_{\sigma, v} m).
\end{aligned}$$

Hence in order to show that for any $a \in A$, $x, y \in X$, $h, l, m \in H$,

$$\langle (a \rtimes_{\rho, u} h)(x \rtimes_{\lambda} l), y \rtimes_{\lambda} m \rangle_{B \rtimes_{\sigma, v} H} = \langle x \rtimes_{\lambda} l, (y \rtimes_{\lambda} m)(a \rtimes_{\rho, u} h)^* \rangle_{B \rtimes_{\sigma, v} H},$$

we have only to show that for any $a \in A$, $x, y \in X$, $h \in H$,

$$\langle (a \rtimes_{\rho, u} h)(x \rtimes_{\lambda} 1), y \rtimes_{\lambda} 1 \rangle_{B \rtimes_{\sigma, v} H} = \langle x \rtimes_{\lambda} 1, (a \rtimes_{\rho, u} h)^*(y \rtimes_{\lambda} 1) \rangle_{B \rtimes_{\sigma, v} H}.$$

For any $a \in A$, $x, y \in X$,

$$\begin{aligned}
& \langle (a \rtimes_{\rho, u} 1)(x \rtimes_{\lambda} 1), y \rtimes_{\lambda} 1 \rangle_{B \rtimes_{\sigma, v} H} = \langle ax \rtimes_{\lambda} 1, y \rtimes_{\lambda} 1 \rangle_{B \rtimes_{\sigma, v} H} \\
&= \langle ax, y \rangle_B = \langle x \rtimes_{\lambda} 1, (a \rtimes_{\rho, u} 1)^*(y \rtimes_{\lambda} 1) \rangle_{B \rtimes_{\sigma, v} H}.
\end{aligned}$$

Also, for any $x, y \in X$, $h \in H$,

$$\begin{aligned}
& \langle (1 \rtimes_{\rho, u} h)(x \rtimes_{\lambda} 1), y \rtimes_{\lambda} 1 \rangle_{B \rtimes_{\sigma, v} H} \\
&= \langle [S(h_{(4)}) \cdot_{\lambda} [h_{(1)} \cdot_{\lambda} x]] \widehat{v}(S(h_{(3)}), h_{(2)}), [h_{(5)}^* \cdot_{\lambda} y] \rangle_B \rtimes_{\sigma, v} h_{(6)}^* \\
&= \langle \widehat{u}(S(h_{(2)}), h_{(1)})x, [h_{(3)}^* \cdot_{\lambda} y] \rangle_B \rtimes_{\sigma, v} h_{(4)}^* \\
&= \langle x \rtimes_{\lambda} 1, \widehat{u}(S(h_{(2)}), h_{(1)})^*[h_{(3)}^* \cdot_{\lambda} y] \rtimes_{\lambda} h_{(4)}^* \rangle_{B \rtimes_{\sigma, v} H} \\
&= \langle x \rtimes_{\lambda} 1, (1 \rtimes_{\rho, u} h)^*(y \rtimes_{\lambda} 1) \rangle_{B \rtimes_{\sigma, v} H}.
\end{aligned}$$

Thus Condition (8) in [9, Lemma 1.3] is satisfied. Moreover, for any $a \in A$, $b \in B$, $x \in X$, $h, l, m \in H$,

$$\begin{aligned}
& (a \rtimes_{\rho, u} h)[(x \rtimes_{\lambda} l)(b \rtimes_{\sigma, v} m)] \\
&= a[h_{(1)} \cdot_{\lambda} x][h_{(2)} \cdot_{\sigma, v} [l_{(1)} \cdot_{\sigma, v} b]] \widehat{v}(h_{(3)}, l_{(2)}) \widehat{v}(h_{(4)} l_{(3)}, m_{(1)}) \rtimes_{\lambda} h_{(5)} l_{(4)} m_{(2)} \\
&= a[h_{(1)} \cdot_{\lambda} x] \widehat{v}(h_{(2)}, l_{(1)}) [h_{(3)} l_{(2)} \cdot_{\sigma, v} b] \widehat{v}(h_{(4)} l_{(3)}, m_{(1)}) \rtimes_{\lambda} h_{(5)} l_{(4)} m_{(2)} \\
&= [(a \rtimes_{\rho, u} h)(x \rtimes_{\lambda} l)](b \rtimes_{\sigma, v} m).
\end{aligned}$$

Thus Condition (7) in [9, Lemma 1.3] is satisfied. Since X is of finite type, there are finite subsets $\{w_i\}_{i=1}^n$ and $\{z_j\}_{j=1}^m$ in X such that

$$x = \sum_{i=1}^n w_i \langle w_i, x \rangle_B = \sum_{j=1}^m A \langle x, z_j \rangle z_j$$

for any $x \in X$. Then we have the following lemma:

Lemma 4.1. *With the above notations, if $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a covariant system, then for any $x \in X$, $h \in H$,*

$$\begin{aligned}
x \rtimes_{\lambda} h &= \sum_{i=1}^n (w_i \rtimes_{\lambda} 1) \langle w_i \rtimes_{\lambda} 1, x \rtimes_{\lambda} h \rangle_{B \rtimes_{\sigma} H} \\
&= \sum_{j=1}^m A \rtimes_{\rho} H \langle x \rtimes_{\lambda} h, z_j \rtimes_{\lambda} 1 \rangle (z_j \rtimes_{\lambda} 1).
\end{aligned}$$

Proof. For any $x \in X$, $h \in H$,

$$\sum_{i=1}^n (w_i \rtimes_{\lambda} 1) \langle w_i \rtimes_{\lambda} 1, x \rtimes_{\lambda} h \rangle_{B \rtimes_{\sigma} H} = \sum_{i=1}^n w_i \langle w_i, x \rangle_B \rtimes_{\lambda} h = x \rtimes_{\lambda} h.$$

Also,

$$\begin{aligned}
& \sum_{j=1}^m {}_{A \rtimes_{\rho} H} \langle x \rtimes_{\lambda} h, z_j \rtimes_{\lambda} 1 \rangle (z_j \rtimes_{\lambda} 1) \\
&= \sum_{j=1}^m {}_A \langle [h_{(2)} S(h_{(1)}) \cdot_{\lambda} x], [S(h_{(3)})^* \cdot_{\lambda} z_j] \rangle [h_{(4)} \cdot_{\lambda} z_j] \rtimes_{\lambda} h_{(5)} \\
&= \sum_{j=1}^m [h_{(2)} \cdot_{\lambda} {}_A \langle [S(h_{(1)}) \cdot_{\lambda} x], z_j \rangle z_j] \rtimes_{\lambda} h_{(3)} \\
&= [h_{(2)} \cdot_{\lambda} [S(h_{(1)}) \cdot_{\lambda} x]] \rtimes_{\lambda} h_{(3)} = x \rtimes_{\lambda} h.
\end{aligned}$$

Therefore, we obtain the conclusion. \square

For any Hilbert C^* -bimodule Y , $l - \text{Ind}[Y]$ and $r - \text{Ind}[Y]$ denote its left index and right index, respectively.

Corollary 4.2. *With the above notations and assumptions,*

$$l - \text{Ind}[X \rtimes_{\lambda} H] = l - \text{Ind}[X] \rtimes_{\sigma} 1, \quad r - \text{Ind}[X \rtimes_{\lambda} H] = r - \text{Ind}[X] \rtimes_{\rho} 1.$$

Proof. By the definitions of the left index and the right index of a Hilbert C^* -bimodule,

$$\begin{aligned}
l - \text{Ind}[X \rtimes_{\lambda} H] &= \sum_{j=1}^m \langle z_j, z_j \rangle_B \rtimes_{\sigma} 1 = l - \text{Ind}[X] \rtimes_{\sigma} 1, \\
r - \text{Ind}[X \rtimes_{\lambda} H] &= \sum_{i=1}^n {}_A \langle w_i, w_i \rangle \rtimes_{\rho} 1 = r - \text{Ind}[X] \rtimes_{\rho} 1.
\end{aligned}$$

\square

Proposition 4.3. *With the above notations and assumptions, $X \rtimes_{\lambda} H$ is a Hilbert $A \rtimes_{\rho} H - B \rtimes_{\sigma} H$ -bimodule of finite type with*

$$l - \text{Ind}[X \rtimes_{\lambda} H] = l - \text{Ind}[X] \rtimes_{\sigma} 1, \quad r - \text{Ind}[X \rtimes_{\lambda} H] = r - \text{Ind}[X] \rtimes_{\rho} 1.$$

Proof. This is immediate by Lemma 4.1, Corollary 4.2 and [9, Lemma 1.3]. \square

Lemma 4.4. *With the above notations, if $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ is a twisted covariant system and X is an $A - B$ -equivalence bimodule, then for any $x \in X$, $h \in H$,*

$$\begin{aligned}
x \rtimes_{\lambda} h &= \sum_{i=1}^n (w_i \rtimes_{\lambda} 1) \langle w_i \rtimes_{\lambda} 1, x \rtimes_{\lambda} h \rangle_{B \rtimes_{\sigma, z} H} \\
&= \sum_{j=1}^m {}_{A \rtimes_{\rho, w} H} \langle x \rtimes_{\lambda} h, z_j \rtimes_{\lambda} 1 \rangle (z_j \rtimes_{\lambda} 1).
\end{aligned}$$

Proof. For any $x \in X$, $h \in H$,

$$\sum_{i=1}^n (w_i \rtimes_\lambda 1) \langle w_i \rtimes_\lambda 1, x \rtimes_\lambda h \rangle_{B \rtimes_{\sigma,v} H} = \sum_{i=1}^n w_i \langle w_i, x \rangle_B \rtimes_\lambda h = x \rtimes_\lambda h.$$

Also,

$$\begin{aligned} & \sum_{j=1}^m {}_{A \rtimes_{\rho,u} H} \langle x \rtimes_\lambda h, z_j \rtimes_\lambda 1 \rangle (z_j \rtimes_\lambda 1) \\ &= \sum_{j=1}^m {}_A \langle x, [S(h_{(1)})^* \cdot_\lambda z_j] \rangle [h_{(2)} \cdot_\lambda z_j] \rtimes_\lambda h_{(3)} \\ &= \sum_{j=1}^m x \langle [S(h_{(1)})^* \cdot_\lambda z_j], [h_{(2)} \cdot_\lambda z_j] \rangle_B \rtimes_\lambda h_{(3)} \\ &= \sum_{j=1}^m x [h_{(1)} \cdot_\sigma \langle z_j, z_j \rangle_B] \rtimes_\lambda h_{(2)} = x \rtimes_\lambda h. \end{aligned}$$

Therefore, we obtain the conclusion. \square

Lemma 4.5. *With the above notations and assumptions, if X is an $A - B$ -equivalence bimodule, then the Hilbert $A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H$ -bimodule is full with the both-sided inner products.*

Proof. For any $x, y \in X$, ${}_{A \rtimes_{\rho,u} H} \langle x \rtimes_\lambda 1, y \rtimes_\lambda 1 \rangle = {}_A \langle x, y \rangle \rtimes_{\rho,u} 1$. Since ${}_{A \rtimes_{\rho,u} H} \langle X \rtimes_\lambda H, X \rtimes_\lambda H \rangle$ is a closed ideal of $A \rtimes_{\rho,u} H$, for any $x, y \in X$, $h \in H$,

$$({}_A \langle x, y \rangle \rtimes_{\rho,u} 1) (1 \rtimes_{\rho,u} h) = {}_A \langle x, y \rangle \rtimes_{\rho,u} h \in {}_{A \rtimes_{\rho,u} H} \langle X \rtimes_\lambda H, X \rtimes_\lambda H \rangle.$$

Since ${}_A \langle X, X \rangle = A$, we obtain that

$${}_{A \rtimes_{\rho,u} H} \langle X \rtimes_\lambda H, X \rtimes_\lambda H \rangle = A \rtimes_{\rho,u} H.$$

Also, for any $x, y \in X$, $h \in H$,

$$\langle x \rtimes_\lambda 1, y \rtimes_\lambda h \rangle_{B \rtimes_{\sigma,v} H} = \langle x, y \rangle_B \rtimes_{\sigma,v} h \in \langle X \rtimes_\lambda H, X \rtimes_\lambda H \rangle_{B \rtimes_{\sigma,v} H}.$$

Since $\langle X, X \rangle_B = B$, we obtain that

$$\langle X \rtimes_\lambda H, X \rtimes_\lambda H \rangle_{B \rtimes_{\sigma,v} H} = B \rtimes_{\sigma,v} H.$$

\square

Corollary 4.6. *With the above notations and assumptions, we suppose that X is an $A - B$ -equivalence bimodule. Then the $X \rtimes_\lambda H$ is an $A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H$ -equivalence bimodule.*

Proof. By Lemma 4.5, it suffices to show that

$$A \rtimes_{\rho,u} H \langle x \rtimes_{\lambda} h, y \rtimes_{\lambda} l \rangle (z \rtimes_{\lambda} m) = (x \rtimes_{\lambda} h) \langle y \rtimes_{\lambda} l, z \rtimes_{\lambda} m \rangle_{B \rtimes_{\sigma,v} H}$$

for any $x, y, z \in X$, $h, l, m \in H$. Since X is an $A - B$ -equivalence bimodule,

$$\begin{aligned} & (x \rtimes_{\lambda} h) \langle y \rtimes_{\lambda} l, z \rtimes_{\lambda} m \rangle_{B \rtimes_{\sigma,v} H} \\ &= x [h_{(1)} \cdot_{\sigma,v} \widehat{v}^*(l_{(2)}^*, S(l_{(1)}^*)) [l_{(3)}^* \cdot_{\sigma,v} \langle y, z \rangle_B] \widehat{v}(l_{(4)}^*, m_{(1)})] \widehat{v}(h_{(2)}, l_{(5)}^* m_{(2)}) \\ & \rtimes_{\lambda} h_{(3)} l_{(6)}^* m_{(3)} \\ &= x [h_{(1)} \cdot_{\sigma,v} \widehat{v}^*(l_{(2)}^*, S(l_{(1)}^*))] [h_{(2)} \cdot_{\sigma,v} [l_{(3)}^* \cdot_{\sigma,v} \langle y, z \rangle_B]] \\ & \times [h_{(3)} \cdot_{\sigma,v} \widehat{v}(l_{(4)}^*, m_{(1)})] \widehat{v}(h_{(4)}, l_{(5)}^* m_{(2)}) \rtimes_{\lambda} h_{(5)} l_{(6)}^* m_{(3)} \\ &= x [h_{(1)} \cdot_{\sigma,v} \widehat{v}^*(l_{(2)}^*, S(l_{(1)}^*))] \widehat{v}(h_{(2)}, l_{(3)}^*) [h_{(3)} l_{(4)}^* \cdot_{\sigma,v} \langle y, z \rangle_B] \\ & \times \widehat{v}(h_{(4)} l_{(5)}^*, m_{(1)}) \rtimes_{\lambda} h_{(5)} l_{(6)}^* m_{(2)} \\ &= x \widehat{v}^*(h_{(1)} l_{(2)}^*, S(l_{(1)}^*)) [h_{(2)} l_{(3)}^* \cdot_{\sigma,v} \langle y, z \rangle_B] \widehat{v}(h_{(3)} l_{(4)}^*, m_{(1)}) \rtimes_{\lambda} h_{(4)} l_{(5)}^* m_{(2)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & A \rtimes_{\rho,u} H \langle x \rtimes_{\lambda} h, y \rtimes_{\lambda} l \rangle (z \rtimes_{\lambda} m) \\ &= A \langle x, [S(h_{(2)} l_{(3)}^*)^* \cdot_{\lambda} y] \widehat{v}(S(h_{(1)} l_{(2)}^*)^*, l_{(1)}) [h_{(3)} l_{(4)}^* \cdot_{\lambda} z] \widehat{v}(h_{(4)} l_{(5)}^*, m_{(1)}) \\ & \rtimes_{\lambda} h_{(5)} l_{(6)}^* m_{(2)} \\ &= x \langle [S(h_{(2)} l_{(3)}^*)^* \cdot_{\lambda} y] \widehat{v}(S(h_{(1)} l_{(2)}^*)^*, l_{(1)}), [h_{(3)} l_{(4)}^* \cdot_{\lambda} z] \rangle_B \widehat{v}(h_{(4)} l_{(5)}^*, m_{(1)}) \\ & \rtimes_{\lambda} h_{(5)} l_{(6)}^* m_{(2)} \\ &= x \widehat{v}^*(h_{(1)} l_{(2)}^*, S(l_{(1)}^*)) [h_{(2)} l_{(3)}^* \cdot_{\sigma,v} \langle y, z \rangle_B] \widehat{v}(h_{(3)} l_{(4)}^*, m_{(1)}) \rtimes_{\lambda} h_{(4)} l_{(5)}^* m_{(2)}. \end{aligned}$$

Therefore, we obtain the conclusion. \square

By the above discussions, we obtain the following:

Corollary 4.7. (1) Let $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ be a twisted covariant system. We suppose that X is an $A - B$ -equivalent bimodule. Then the crossed product $X \rtimes_{\lambda} H$ is an $A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H$ -equivalence bimodule.

(2) Let $(A, B, X, \rho, \sigma, \lambda, H^0)$ be a covariant system. Then the crossed $X \rtimes_{\lambda} H$ is a Hilbert $A \rtimes_{\rho} H - B \rtimes_{\sigma} H$ -bimodule of finite type.

In the situation of Corollary 4.7(1), let $X \rtimes_{\lambda} H$ be the crossed product associated to a twisted covariant system $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$, where X is an $A - B$ -equivalence bimodule. Then we define the dual covariant system with $X \rtimes_{\lambda} H$ as follows: Let $\widehat{\rho}$ and $\widehat{\sigma}$ be the dual coactions of H on $A \rtimes_{\rho,u} H$ and $B \rtimes_{\sigma,v} H$ of (ρ, u) and (σ, v) , respectively. Let $\widehat{\lambda}$ be the dual coaction of H on $X \rtimes_{\lambda} H$ defined by

$$\widehat{\lambda}(x \rtimes_{\lambda} h) = (x \rtimes_{\lambda} h_{(1)}) \otimes h_{(2)}$$

for any $x \in X$, $h \in H$. Then by easy computations, we can see that

$$(A \rtimes_{\rho,u} H, B \rtimes_{\sigma,v} H, X \rtimes_{\lambda} H, \widehat{\rho}, \widehat{\sigma}, \widehat{\lambda}, H)$$

is a covariant system. Hence we obtain the following:

Corollary 4.8. *Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B , respectively. Then the following conditions are equivalent:*

- (1) *The twisted coaction (ρ, u) is strongly Morita equivalent to the twisted coaction (σ, v) ,*
- (2) *The dual coaction $\widehat{\rho}$ of (ρ, u) is strongly Morita equivalent to the dual coaction $\widehat{\sigma}$ of (σ, v) .*

Proof. By the above discussion, it is clear that Condition (1) implies Condition (2). We suppose Condition (2). Then by Condition (2), we can see that $\widehat{\rho}$ is strongly Morita equivalent to $\widehat{\sigma}$, where $\widehat{\rho}$ and $\widehat{\sigma}$ are the dual coactions of ρ and σ , respectively. By [11, Theorem 3.3], there is an isomorphism Ψ of $M_N(A)$ onto $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ such that $\widehat{\rho}$ is exterior equivalent to the twisted coaction

$$((\Psi \otimes \text{id}) \circ (\rho \otimes \text{id}) \circ \Psi^{-1}, (\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N)).$$

Hence by Lemma 3.12, $\widehat{\rho}$ is strongly Morita equivalent to $(\rho \otimes \text{id}, u \otimes I_N)$. Thus, by Lemma 3.16 and Corollary 3.8, $\widehat{\rho}$ is strongly Morita equivalent to (ρ, u) . Similarly $\widehat{\sigma}$ is strongly Morita equivalent to (σ, v) . Therefore, by Corollary 3.8, (ρ, u) is strongly Morita equivalent to (σ, v) . \square

Also, in the situation of Corollary 4.7(2), we can see that

$$(A \rtimes_{\rho} H, B \rtimes_{\sigma} H, X \rtimes_{\lambda} H, \widehat{\rho}, \widehat{\sigma}, \widehat{\lambda}, H)$$

is a covariant system in the same way as above.

5. DUALITY

In this section, we present a duality theorem for a crossed product of a Hilbert C^* -bimodule of finite type by a (twisted) coaction of a finite dimensional C^* -Hopf algebra in the same way as in [11]. As mentioned in Section 1, Guo and Zhang have already obtained a duality result using the language of multiplicative unitary elements and Kac systems in [5]. But we give our duality result because our approach to coactions of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra is a useful addition to the main result in Section 6. First, we suppose Condition (1) or Condition (2) in Corollary 4.7. In the both cases, we can consider the dual covariant systems

$$(A \rtimes_{\rho,u} H, B \rtimes_{\sigma,v} H, X \rtimes_{\lambda} H, \widehat{\rho}, \widehat{\sigma}, \widehat{\lambda}, H),$$

$$(A \rtimes_{\rho} H, B \rtimes_{\sigma} H, X \rtimes_{\lambda} H, \widehat{\rho}, \widehat{\sigma}, \widehat{\lambda}, H).$$

Let Λ be the set of all triplets (i, j, k) where $i, j = 1, 2, \dots, d_k$ and $k = 1, 2, \dots, K$ and $\sum_{k=1}^K d_k^2 = N$. For each $I = (i, j, k) \in \Lambda$, let W_I^{ρ}, V_I^{ρ} be elements in $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ defined by

$$W_I^{\rho} = \sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k, \quad V_I^{\rho} = (1 \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} \tau)(W_I^{\rho} \rtimes_{\widehat{\rho}} 1^0).$$

Similarly for each $I = (i, j, k) \in \Lambda$, we define elements

$$W_I^{\sigma} = \sqrt{d_k} \rtimes_{\sigma, v} w_{ij}^k, \quad V_I^{\sigma} = (1 \rtimes_{\sigma, v} 1 \rtimes_{\widehat{\sigma}} \tau)(W_I^{\sigma} \rtimes_{\widehat{\sigma}} 1^0)$$

in $B \rtimes_{\sigma, v} H \rtimes_{\widehat{\sigma}} H^0$. We regard $M_N(\mathbf{C})$ as a Hilbert $M_N(\mathbf{C}) - M_N(\mathbf{C})$ -bimodule in the usual way. Let $X \otimes M_N(\mathbf{C})$ be an exterior tensor product of X and $M_N(\mathbf{C})$, which is a Hilbert $A \otimes M_N(\mathbf{C}) - B \otimes M_N(\mathbf{C})$ -bimodule. In the same way as Lemma 2.1, we can see that $X \otimes M_N(\mathbf{C})$ is of finite type. Let $\{f_{IJ}\}_{I, J \in \Lambda}$ be a system of matrix units of $M_N(\mathbf{C})$. Let Ψ_X be a linear map from $X \otimes M_N(\mathbf{C})$ to $X \rtimes_{\lambda} H \rtimes_{\widehat{\lambda}} H^0$ defined by

$$\Psi_X\left(\sum_{I, J} x_{IJ} \otimes f_{IJ}\right) = \sum_{I, J} V_I^{\rho*}(x_{IJ} \rtimes_{\lambda} 1 \rtimes_{\widehat{\lambda}} 1^0) V_J^{\sigma}$$

for any $x_{IJ} \in X$. Let Ψ_A and Ψ_B be isomorphisms of $A \otimes M_N(\mathbf{C})$ and $B \otimes M_N(\mathbf{C})$ onto $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ and $B \rtimes_{\sigma, v} H \rtimes_{\widehat{\sigma}} H^0$ defined by

$$\begin{aligned} \Psi_A\left(\sum_{I, J} a_{IJ} \otimes f_{IJ}\right) &= \sum_{I, J} V_I^{\rho*}(a_{IJ} \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J^{\rho}, \\ \Psi_B\left(\sum_{I, J} b_{IJ} \otimes f_{IJ}\right) &= \sum_{I, J} V_I^{\sigma*}(a_{IJ} \rtimes_{\sigma, v} 1 \rtimes_{\widehat{\sigma}} 1^0) V_J^{\sigma} \end{aligned}$$

for any $a_{IJ} \in A, b_{IJ} \in B$, respectively (see [11]).

Lemma 5.1. *With the above notations,*

$$\begin{aligned}
(1) & \Psi_X((\sum_{I,J} a_{IJ} \otimes f_{IJ})(\sum_{I,J} x_{IJ} \otimes f_{IJ})) \\
&= \Psi_A(\sum_{I,J} a_{IJ} \otimes f_{IJ}) \Psi_X(\sum_{I,J} x_{IJ} \otimes f_{IJ}), \\
(2) & \Psi_X((\sum_{I,J} x_{IJ} \otimes f_{IJ})(\sum_{I,J} b_{IJ} \otimes f_{IJ})) \\
&= \Psi_X(\sum_{I,J} x_{IJ} \otimes f_{IJ}) \Psi_B(\sum_{I,J} b_{IJ} \otimes f_{IJ}), \\
(3) & {}_{A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0} \langle \Psi_X(\sum_{I,J} x_{IJ} \otimes f_{IJ}), \Psi_X(\sum_{I,J} y_{IJ} \otimes f_{IJ}) \rangle \\
&= \Psi_A({}_{A \otimes M_N(\mathbf{C})} \langle \sum_{I,J} x_{IJ} \otimes f_{IJ}, \sum_{I,J} y_{IJ} \otimes f_{IJ} \rangle), \\
(4) & \langle \Psi_X(\sum_{I,J} x_{IJ} \otimes f_{IJ}), \Psi_X(\sum_{I,J} y_{IJ} \otimes f_{IJ}) \rangle_{B \rtimes_{\sigma,v} H \rtimes_{\hat{\sigma}} H^0} \\
&= \Psi_B(\langle \sum_{I,J} x_{IJ} \otimes f_{IJ}, \sum_{I,J} y_{IJ} \otimes f_{IJ} \rangle_{B \otimes M_N(\mathbf{C})})
\end{aligned}$$

for any $a_{IJ} \in A$, $b_{IJ} \in B$, $x_{IJ}, y_{IJ} \in X$, $I, J \in \Lambda$.

Proof. This is immediate by routine computations. Indeed,

$$\Psi_X((\sum_{I,J} a_{IJ} \otimes f_{IJ})(\sum_{I,J} x_{IJ} \otimes f_{IJ})) = \sum_{I,J,L} V_I^{\rho*}(a_{IL} x_{LJ} \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} 1^0) V_J^{\sigma}.$$

On the other hand, by [11, Lemma 3.1]

$$\begin{aligned}
& \Psi_A(\sum_{I,J} a_{IJ} \otimes f_{IJ}) \Psi_X(\sum_{L,M} x_{LM} \otimes f_{LM}) \\
&= \sum_{I,J,M} V_I^{\rho*}(a_{IJ} \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} 1^0)(1 \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} \tau)(x_{JM} \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} 1^0) V_M^{\sigma} \\
&= \sum_{I,J,M} V_I^{\rho*}(a_{IJ} x_{JM} \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} 1^0) V_M^{\sigma}.
\end{aligned}$$

Thus we obtain Equation (1). Similarly we can obtain the Equation (2). Also, by [11, Lemma 3.1]

$$\begin{aligned}
& A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0 \langle \Psi_X(\sum_{I,J} x_{IJ} \otimes f_{IJ}), \Psi_X(\sum_{I,J} y_{IJ} \otimes f_{IJ}) \rangle \\
&= \sum_{I,J,I_1,J_1} A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0 \langle V_I^{\rho*}(x_{IJ} \rtimes_{\lambda} 1 \rtimes_{\widehat{\lambda}} 1^0) V_J^{\sigma}, V_{I_1}^{\rho*}(y_{I_1 J_1} \rtimes_{\lambda} 1 \rtimes_{\widehat{\lambda}} 1^0) V_{J_1}^{\sigma} \rangle \\
&= \sum_{I,J,I_1} V_I^{\rho*}(A \rtimes_{\rho,u} H \langle x_{IJ} \rtimes_{\lambda} 1, y_{I_1 J} \rtimes_{\lambda} 1 \rangle \rtimes_{\widehat{\rho}} \tau) V_{I_1}^{\rho} \\
&= \sum_{I,J,I_1} V_I^{\rho*}(A \langle x_{IJ}, y_{I_1 J} \rangle \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_{I_1}^{\rho}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \Psi_A(A \otimes_{M_N(\mathbf{C})} \langle \sum_{I,J} x_{IJ} \otimes f_{IJ}, \sum_{I_1,J_1} y_{I_1 J_1} \otimes f_{I_1 J_1} \rangle) \\
&= \sum_{I,J,I_1} \Psi_A(A \langle x_{IJ}, y_{I_1 J} \rangle \otimes f_{I_1 J_1}) \\
&= \sum_{I,J,I_1} V_I^{\rho*}(A \langle x_{IJ}, y_{I_1 J} \rangle \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} 1^0) V_{I_1}^{\rho}.
\end{aligned}$$

Thus we obtain Equation (3). Furthermore,

$$\begin{aligned}
& \langle \Psi_X(\sum_{I,J} x_{IJ} \otimes f_{IJ}), \Psi_X(\sum_{I_1,J_1} y_{I_1 J_1} \otimes f_{I_1 J_1}) \rangle_{B \rtimes_{\sigma,v} H \rtimes_{\widehat{\sigma}} H^0} \\
&= \sum_{I,J,J_1} V_J^{\sigma*}(\langle x_{IJ}, y_{I_1 J_1} \rangle_B \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} 1^0) V_{J_1}^{\sigma} \\
&= \Psi_B(\langle \sum_{I,J} x_{IJ} \otimes f_{IJ}, \sum_{I_1,J_1} y_{I_1 J_1} \otimes f_{I_1 J_1} \rangle_{B \otimes_{M_N(\mathbf{C})}}).
\end{aligned}$$

Thus we obtain Equation (4). \square

By the above lemma, we can see that Ψ_X is injective. Next, we show that Ψ_X is surjective.

Lemma 5.2. *With the above notations,*

$$(X \rtimes_{\lambda} H \rtimes_{\widehat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau)(B \rtimes_{\sigma,v} H \rtimes_{\widehat{\sigma}} 1^0) = X \rtimes_{\lambda} H \rtimes_{\widehat{\lambda}} H^0.$$

Proof. Let $x \in X$, $h \in H$, $\phi \in H^0$. Since

$$\sum_{i,j,k} (\sqrt{d_k} \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0)^* (1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau) (\sqrt{d_k} \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0) = 1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} 1^0$$

by [10, Proposition 3.18],

$$\begin{aligned}
& x \rtimes_{\lambda} h \rtimes_{\widehat{\lambda}} \phi \\
&= \sum_{i,j,k} (x \rtimes_{\lambda} h \rtimes_{\widehat{\lambda}} \phi) (\sqrt{d_k} \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0)^* (1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau) \\
&\times (\sqrt{d_k} \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0) \\
&= \sum_{i,j,k,j_1,j_2} d_k ((x \rtimes_{\lambda} h) [\phi \cdot_{\widehat{\sigma}} (\widehat{v}(S(w_{j_1j_2}^k), w_{ij_1}^k)^* \rtimes_{\sigma,v} w_{j_2j}^{k*})] \rtimes_{\widehat{\lambda}} \tau) \\
&\times (1 \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0) \\
&= \sum_{i,j,k,j_1,j_2,j_3} d_k \phi(w_{j_3j}^{k*}) ((x \rtimes_{\lambda} h) (\widehat{v}(S(w_{j_1j_2}^k), w_{ij_1}^k)^* \rtimes_{\sigma,v} w_{j_2j_3}^{k*}) \rtimes_{\widehat{\lambda}} 1^0) \\
&\times (1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau) (1 \rtimes_{\sigma,v} w_{ij}^k \rtimes_{\widehat{\sigma}} 1^0).
\end{aligned}$$

Therefore we obtain the conclusion. \square

Let E_1^{σ} be the canonical conditional expectation from $B \rtimes_{\sigma,v} H$ onto B defined by $E_1^{\sigma}(b \rtimes_{\sigma,v} h) = \tau(h)b$ for any $b \in B$, $h \in H$. Let E_1^{λ} be a linear map from $X \rtimes_{\lambda} H$ onto X defined by

$$E_1^{\lambda}(x \rtimes_{\lambda} h) = \tau(h)x$$

for any $x \in X$, $h \in H$.

Lemma 5.3. *With the above notations, for any $x \in X$, $h \in H$,*

$$\sum_{i,j,k} (\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)^* E_1^{\lambda}((\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)(x \rtimes_{\lambda} h)) = x \rtimes_{\lambda} h.$$

Proof. This is also immediate by routine computations. Indeed, for any $x \in X$, $h \in H$, by [17, Theorem 2.2],

$$\begin{aligned}
& \sum_{i,j,k} (\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)^* E_1^{\lambda}((\sqrt{d_k} \rtimes_{\rho,u} w_{ij}^k)(x \rtimes_{\lambda} h)) \\
&= \sum_{i,j,k,j_1,j_2,s,s_1,s_2,s_3} d_k \widehat{u}^*(w_{ss_1}^{k*}, w_{si}^k) [w_{s_1s_2}^{k*} \cdot_{\lambda} [w_{ij_1}^k \cdot_{\lambda} x]] [w_{s_2s_3}^{k*} \cdot_{\sigma,v} \widehat{v}(w_{j_1j_2}^k, h_{(1)})] \\
&\rtimes_{\lambda} \tau(w_{j_2j}^k h_{(2)}) w_{s_3j}^{k*} \\
&= \sum_{i,j,k,j_1,j_2,s_2,s_3} d_k x \widehat{v}^*(w_{is_2}^{k*}, w_{ij_1}^k) [w_{s_2s_3}^{k*} \cdot_{\sigma,v} \widehat{v}(w_{j_1j_2}^k, h_{(1)})] \rtimes_{\lambda} \tau(w_{j_2j}^k h_{(2)}) w_{s_3j}^{k*} \\
&= \sum_{i,j,k,j_1,j_2,s_2,s_3} d_k x \widehat{v}(w_{is_2}^{k*} w_{ij_1}^k, h_{(1)}) \widehat{v}^*(w_{s_2s_3}^{k*}, w_{j_1j_2}^k h_{(2)}) \tau(w_{j_2j}^k h_{(3)}) \rtimes_{\lambda} w_{s_3j}^{k*} \\
&= \sum_{j,k,s_2} d_k x \tau(w_{s_2j}^k h) \rtimes_{\lambda} S(w_{js_2}^k) = \sum_{i,k,s_2} d_k x \tau(w_{s_2j}^k h_{(1)}) \rtimes_{\lambda} S(w_{js_2}^k h_{(2)}) S(h_{(3)}) \\
&= x \rtimes_{\lambda} S(\tau(Neh_{(1)})S(h_{(2)})) = x \rtimes_{\lambda} h.
\end{aligned}$$

Therefore, we obtain the conclusion. \square

Lemma 5.4. *With the above notations,*

$$(1 \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} \phi)(x \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} 1^0) = x \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} \phi = (x \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\hat{\sigma}} \phi)$$

for any $x \in X$, $\phi \in H^0$.

Proof. For any $x \in X$, $\phi \in H^0$,

$$\begin{aligned} (1 \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} \phi)(x \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} 1^0) &= [\phi_{(1)} \cdot_{\hat{\lambda}} (x \rtimes_{\lambda} 1)] \rtimes_{\hat{\lambda}} \phi_{(2)} = x \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} \phi \\ &= (x \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\hat{\sigma}} \phi). \end{aligned}$$

□

Lemma 5.5. *With the above notations, Ψ_X is surjective.*

Proof. By Lemma 5.2, it suffices to show that for any $b \in B$, $x \in X$, $h, l \in H$, there is an element $y \in X \otimes M_N(\mathbf{C})$ such that

$$\Psi_X(y) = (x \rtimes_{\lambda} h \rtimes_{\hat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\hat{\sigma}} \tau)(b \rtimes_{\sigma,v} l \rtimes_{\hat{\sigma}} 1^0).$$

By Lemma 5.3 and [10, Proposition 3.18]

$$\begin{aligned} x \rtimes_{\lambda} h &= \sum_I W_I^{\rho*}(E_1^{\lambda}(W_I^{\rho}(x \rtimes_{\lambda} h)) \rtimes_{\lambda} 1), \\ b \rtimes_{\sigma,v} l &= \sum_I (E_1^{\sigma}((b \rtimes_{\sigma,v} l)W_I^{\sigma*}) \rtimes_{\sigma,v} 1)W_I^{\sigma}. \end{aligned}$$

Thus

$$\begin{aligned} &(x \rtimes_{\lambda} h \rtimes_{\hat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\hat{\sigma}} \tau)(b \rtimes_{\sigma,v} l \rtimes_{\hat{\sigma}} 1^0) \\ &= \sum_{I,J} (W_I^{\rho*} \rtimes_{\hat{\rho}} 1^0)(E_1^{\lambda}(W_I^{\rho}(x \rtimes_{\lambda} h)) \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} \tau) \\ &\quad \times (E_1^{\sigma}((b \rtimes_{\sigma,v} l)W_J^{\sigma*}) \rtimes_{\sigma,v} 1 \rtimes_{\hat{\sigma}} \tau)(W_J^{\sigma} \rtimes_{\hat{\sigma}} 1^0). \end{aligned}$$

Since

$$E_1^{\lambda}(W_I^{\rho}(x \rtimes_{\lambda} h)) \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} \tau = (1 \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} \tau)(E_1^{\lambda}(W_I^{\rho}(x \rtimes_{\lambda} h)) \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} 1^0)$$

by Lemma 5.4,

$$\begin{aligned} &(x \rtimes_{\lambda} h \rtimes_{\hat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\hat{\sigma}} \tau)(b \rtimes_{\sigma,v} l \rtimes_{\hat{\sigma}} 1^0) \\ &= \sum_{I,J} V_I^{\rho*}[E_1^{\lambda}(W_I^{\rho}(x \rtimes_{\lambda} h))E_1^{\sigma}((b \rtimes_{\sigma,v} l)W_J^{\sigma*}) \rtimes_{\lambda} 1 \rtimes_{\hat{\lambda}} 1^0]V_J^{\sigma}. \end{aligned}$$

Since $E_1^{\lambda}(W_I^{\rho}(x \rtimes_{\lambda} h))E_1^{\sigma}((b \rtimes_{\sigma,v} l)W_J^{\sigma*}) \in X$, we obtain the conclusion. □

Let \widehat{V}^{ρ} be a linear map from H to $A \rtimes_{\rho,u} H$ defined by $\widehat{V}^{\rho}(h) = 1 \rtimes_{\rho,u} h$ for any $h \in H$. By [10], \widehat{V}^{ρ} is a unitary element in $\text{Hom}(H, A \rtimes_{\rho,u} H)$. Let V^{ρ} be the unitary element in $(A \rtimes_{\rho,u} H) \otimes H^0$ induced by \widehat{V}^{ρ} . Similarly, we also define unitary elements $\widehat{V}^{\sigma} \in \text{Hom}(H, B \rtimes_{\sigma,v} H)$ and $V^{\sigma} \in (B \rtimes_{\sigma,v} H) \otimes H^0$.

Lemma 5.6. *With the above notations, for any $x \in X$, $h \in H$,*

$$[h \cdot_\lambda x] \rtimes_\lambda 1 = \widehat{V}^\rho(h_{(1)})(x \rtimes_\lambda 1) \widehat{V}^{\sigma*}(h_{(2)}).$$

Proof. This is also immediate by routine computations. Indeed, for any $x \in X$, $h \in H$,

$$\begin{aligned} & \widehat{V}^\rho(h_{(1)})(x \rtimes_\lambda 1) \widehat{V}^{\sigma*}(h_{(2)}) \\ &= [h_{(1)} \cdot_\lambda x] [h_{(2)} \cdot_{\sigma,v} \widehat{v}^*(S(h_{(7)}), h_{(8)})] \widehat{v}(h_{(3)}, S(h_{(6)})) \rtimes_\lambda h_{(4)} S(h_{(5)}) \\ &= [h_{(1)} \cdot_\lambda x] \widehat{v}(h_{(2)}, S(h_{(5)}) h_{(6)}) \widehat{v}^*(h_{(3)} S(h_{(4)}), h_{(7)}) \rtimes_\lambda 1 = [h \cdot_\lambda x] \rtimes_\lambda 1. \end{aligned}$$

□

Theorem 5.7. (Cf. Guo and Zhang [5, Theorem 2.7]) *Let A, B be unital C^* -algebras and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Then the following hold:*

(1) *If X is an $A - B$ -equivalence bimodule and $(A, B, X, \rho, u, \sigma, v, \lambda, H^0)$ is a twisted covariant system, then there is a linear isomorphism Ψ_X from $X \otimes M_N(\mathbf{C})$ onto $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$ which satisfies Conditions (1)-(4) in Lemma 5.1, where $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$ is an $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0 - B \rtimes_{\sigma,v} H \rtimes_{\widehat{\sigma}} H^0$ -equivalence bimodule and $X \otimes M_N(\mathbf{C})$ is an exterior tensor product of an $A - B$ -equivalence bimodule X and an $M_N(\mathbf{C}) - M_N(\mathbf{C})$ -equivalence bimodule $M_N(\mathbf{C})$. Furthermore, there are unitary elements $U \in (A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ and $V \in (B \rtimes_{\sigma,v} H \rtimes_{\widehat{\sigma}} H^0) \otimes H^0$ such that*

$$\widehat{\lambda}(x)V^* = ((\Psi_X \otimes \text{id}) \circ (\lambda \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi_X^{-1})(x)$$

for any $x \in X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$.

(2) *If X is a Hilbert $A - B$ -bimodule of finite type and $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a covariant system, then there is a linear isomorphism Ψ_X from $X \otimes M_N(\mathbf{C})$ onto $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$ which satisfies Conditions (1)-(4) in Lemma 5.1, where $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$ is a Hilbert $A \rtimes_\rho H \rtimes_{\widehat{\rho}} H^0 - B \rtimes_\sigma H \rtimes_{\widehat{\sigma}} H^0$ -bimodule of finite type and $X \otimes M_N(\mathbf{C})$ is an exterior tensor product of a Hilbert $A - B$ -bimodule X of finite type and an $M_N(\mathbf{C}) - M_N(\mathbf{C})$ -equivalence bimodule $M_N(\mathbf{C})$. Furthermore, there are unitary elements $U \in (A \rtimes_\rho H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ and $V \in (B \rtimes_\sigma H \rtimes_{\widehat{\sigma}} H^0) \otimes H^0$ such that*

$$\widehat{\lambda}(x)V^* = ((\Psi_X \otimes \text{id}) \circ (\lambda \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi_X^{-1})(x)$$

for any $x \in X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$.

Proof. (1) Let Ψ_X be as in Lemma 5.1. By Lemmas 5.1 and 5.5, we can see that Ψ_X is a linear isomorphism from $X \otimes M_N(\mathbf{C})$ onto $X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$. By [11,

Theorem 3.3], there are U and V , unitary elements in $(A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0) \otimes H^0$ and $(B \rtimes_{\sigma,v} H \rtimes_{\hat{\sigma}} H^0) \otimes H^0$ such that

$$\text{Ad}(U) \circ \widehat{\hat{\rho}} = (\Psi_A \otimes \text{id}) \circ (\rho \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi_A^{-1},$$

$$\text{Ad}(V) \circ \widehat{\hat{\sigma}} = (\Psi_B \otimes \text{id}) \circ (\sigma \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi_B^{-1},$$

respectively. Let V^ρ and V^σ be as above. For any $\sum_{I,J} x_{IJ} \otimes f_{IJ} \in X \otimes M_N(\mathbf{C})$,

$$\begin{aligned} & U \widehat{\hat{\lambda}}(\Psi_X(\sum_{I,J} x_{IJ} \otimes f_{IJ})) V^* \\ &= \sum_{I,J} (V_I^{\rho*} \otimes 1^0) V^\rho \widehat{\hat{\lambda}}((1 \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} \tau)(x_{IJ} \rtimes_\lambda 1 \rtimes_{\hat{\lambda}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\hat{\sigma}} \tau)) \\ & \times V^{\sigma*}(V_J^\sigma \otimes 1^0) \end{aligned}$$

by [11, Lemma 3.1] since

$$U = \sum_I (V_I^{\rho*} \otimes 1^0) V^\rho \widehat{\hat{\rho}}(V_I^\rho), \quad V = \sum_I (V_I^{\sigma*} \otimes 1^0) V^\sigma \widehat{\hat{\sigma}}(V_I^\sigma).$$

Since

$$\begin{aligned} \widehat{\hat{\rho}}(1 \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} \tau) &= V^{\rho*}((1 \rtimes_{\rho,u} 1 \rtimes_{\hat{\rho}} \tau) \otimes 1^0) V^\rho, \\ \widehat{\hat{\sigma}}(1 \rtimes_{\sigma,v} 1 \rtimes_{\hat{\sigma}} \tau) &= V^{\sigma*}((1 \rtimes_{\sigma,v} 1 \rtimes_{\hat{\sigma}} \tau) \otimes 1^0) V^\sigma \end{aligned}$$

by the proof of [10, Proposition 3.19],

$$\begin{aligned} & U \widehat{\hat{\lambda}}(\Psi_X(\sum_{I,J} x_{IJ} \otimes f_{IJ})) V^* \\ &= \sum_{I,J} (V_I^{\rho*} \otimes 1^0) V^\rho ((x_{IJ} \rtimes_\lambda 1 \rtimes_{\hat{\lambda}} 1^0) \otimes 1^0) V^{\sigma*}(V_J^\sigma \otimes 1^0) \\ &= \sum_{I,J} (V_I^{\rho*} \otimes 1^0) \lambda(x_{IJ} \rtimes_\lambda 1 \rtimes_{\hat{\lambda}} 1^0) (V_J^\sigma \otimes 1^0) \end{aligned}$$

by Lemma 5.6, where we identify X with $X \rtimes_\lambda 1$ and $X \rtimes_\lambda 1 \rtimes_{\hat{\lambda}} 1^0$. On the other hand,

$$((\Psi_X \otimes \text{id}) \circ (\lambda \otimes \text{id}))(x_{IJ} \otimes f_{IJ}) = (\Psi_X \otimes \text{id})(\lambda(x_{IJ}) \otimes f_{IJ}).$$

We write that $\lambda(x_{IJ}) = \sum_i y_{IJi} \otimes \phi_i$, where $\phi_i \in H^0$, $y_{IJi} \in X$ for any I, J, i . Then

$$\begin{aligned} ((\Psi_X \otimes \text{id}) \circ (\lambda \otimes \text{id}))(x_{IJ} \otimes f_{IJ}) &= \sum_{I,J,i} V_I^{\rho*}(y_{IJi} \rtimes_\lambda 1 \rtimes_{\hat{\lambda}} 1^0) V_J^\sigma \otimes \phi_i \\ &= \sum_{I,J} (V_I^{\rho*} \otimes 1^0) \lambda(x_{IJ} \rtimes_\lambda 1 \rtimes_{\hat{\lambda}} 1^0) (V_J^\sigma \otimes 1^0). \end{aligned}$$

Therefore, we obtain the conclusion.

(2) We can prove (2) in the same way as (1). \square

6. THE STRONG MORITA EQUIVALENCE FOR COACTIONS AND THE ROHLIN PROPERTY

For a unital C^* -algebra A , we set

$$\begin{aligned} c_0(A) &= \{ (a_n) \in l^\infty(\mathbf{N}, A) \mid \lim_{n \rightarrow \infty} \|a_n\| = 0 \}, \\ A^\infty &= l^\infty(\mathbf{N}, A)/c_0(A). \end{aligned}$$

We denote an element in A^∞ by the symbol $[a_n]$ for an element $(a_n) \in l^\infty(\mathbf{N}, A)$. We identify A with the C^* -subalgebra of A^∞ consisting of the equivalence classes of constant sequences and set

$$A_\infty = A^\infty \cap A'.$$

Let X be a Hilbert $A - B$ -bimodule of finite type, where B is a unital C^* -algebra. We define X^∞ in the same way as above. We set

$$\begin{aligned} c_0(X) &= \{ (x_n) \in l^\infty(\mathbf{N}, X) \mid \lim_{n \rightarrow \infty} \|x_n\| = 0 \}, \\ X^\infty &= l^\infty(\mathbf{N}, X)/c_0(X). \end{aligned}$$

We denote an element in X^∞ by the symbol $[x_n]$ for an element $(x_n) \in l^\infty(\mathbf{N}, X)$. We regard X^∞ as an $A^\infty - B^\infty$ -bimodule as follows: For any $[a_n] \in A^\infty$, $[b_n] \in B^\infty$, $[x_n] \in X^\infty$,

$$[a_n][x_n] = [a_n x_n], \quad [x_n][b_n] = [x_n b_n].$$

Also, we define the left A^∞ -valued inner product and the right B^∞ -valued inner product as follows: For any $[x_n], [y_n] \in X^\infty$,

$$A^\infty \langle [x_n], [y_n] \rangle = [A \langle x_n, y_n \rangle], \quad \langle [x_n], [y_n] \rangle_{B^\infty} = [\langle x_n, y_n \rangle_B].$$

By [15, Lemma 2.5] and easy computations, the above definitions are well-defined. We identify X with the Hilbert $A^\infty - B^\infty$ -subbimodule of X^∞ consisting of the equivalence classes of constant sequences. Also, we can see that X^∞ is a complex vector space satisfying Conditions (1)-(8) in [9, Lemma 1.3]. Since X is of finite type, there are finite subsets $\{u_i\}_{i=1}^n, \{v_j\}_{j=1}^m \subset X$ such that for any $x \in X$,

$$\sum_{i=1}^n u_i \langle u_i, x \rangle_B = x = \sum_{j=1}^m A \langle x, v_j \rangle v_j.$$

Then we can regard $u_i, v_j \in X$ as elements in X^∞ for $i = 1, \dots, n, j = 1, \dots, m$. Thus X^∞ is a Hilbert $A^\infty - B^\infty$ -bimodule of finite type by [9, Lemma 1.3]. Furthermore, if X is an $A - B$ -equivalence bimodule, then X^∞ is an $A^\infty - B^\infty$ -equivalence bimodule.

Lemma 6.1. *With the above notations, we suppose that X is an $A - B$ -equivalence bimodule. Let $b \in B^\infty$. If $xb = 0$ for any $x \in X$, then $b = 0$, where we regard X as the Hilbert $A^\infty - B^\infty$ -subbimodule of X^∞ .*

Proof. Since $b \in B^\infty$, we write that $b = [b_m]$, where $b_m \in B$ for any $m \in \mathbf{N}$. Since $xb = 0$, $\|xb_m\| \rightarrow 0$ ($m \rightarrow \infty$). For any $y \in X$,

$$\|\langle y, x \rangle_B b_m\| = \|\langle y, xb_m \rangle_B\| \leq \|y\| \|xb_m\| \rightarrow 0 \text{ } (m \rightarrow \infty)$$

by [15, Lemma 2.5]. On the other hand, there are $x_1, \dots, x_n, y_1, \dots, y_n \in X$ such that $\sum_{i=1}^n \langle y_i, x_i \rangle_B = 1$ since X is full with the right B -valued inner product. Hence

$$\|b_m\| = \left\| \sum_{i=1}^n \langle y_i, x_i \rangle_B b_m \right\| \leq \sum_{i=1}^n \|\langle y_i, x_i \rangle_B b_m\| \rightarrow 0.$$

Therefore $b = 0$. \square

We are in position to present the main result in this paper. Before doing it, we give the definitions of the approximate representability and the Rohlin property for a coaction of a finite dimensional C^* -Hopf algebra on a unital C^* -algebra and a remark on the definitions.

Definition 6.2. (Cf. [11, Definitions 4.3 and 5.1]) Let (ρ, u) be a twisted coaction of a finite dimensional C^* -Hopf algebra H^0 on a unital C^* -algebra A . We say that (ρ, u) is *approximately representable* if there is a unitary element $w \in A^\infty \otimes H^0$ satisfying the following conditions:

- (1) $\rho(a) = (\text{Ad}(w) \circ \rho_{H^0}^A)(a)$ for any $a \in A$,
- (2) $u = (w \otimes 1^0)(\rho_{H^0}^{A^\infty} \otimes \text{id})(w)(\text{id} \otimes \Delta^0)(w^*)$,
- (3) $u = (\rho^\infty \otimes \text{id})(w)(w \otimes 1^0)(\text{id} \otimes \Delta^0)(w^*)$.

Also, we say that (ρ, u) has the *Rohlin property* if its dual coaction $\widehat{\rho}$ of H on $A \rtimes_\rho H$ is approximately representable.

By [11, Corollary 6.4], we can see that a coaction ρ of H^0 on A has the Rohlin property if and only if there is a projection $p \in A_\infty$ such that $e \cdot_{\rho^\infty} p = \frac{1}{N}$, where $N = \dim(H)$.

Theorem 6.3. *Let H be a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let ρ and σ be coactions of H^0 on unital C^* -algebras A and B , respectively. We suppose that ρ is strongly Morita equivalent to σ . Then ρ has the Rohlin property if and only if σ has the Rohlin property.*

Proof. Since ρ and σ are strongly Morita equivalent, there are an $A - B$ -equivalence bimodule X and a coaction λ of H^0 on X with respect to (A, B, ρ, σ) . According to the proof of Rieffel [16, Proposition 2.1], we obtain

the following: Since X is full with the right B -valued inner product, there are elements $x_1, \dots, x_n, y_1, \dots, y_n \in X$ such that $\sum_{i=1}^n \langle x_i, y_i \rangle_B = 1$. Let $E = A \otimes M_n(\mathbf{C})$ and we consider X^n as an $E - B$ -equivalence bimodule in the usual way. Let $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in X^n$. Let $z = {}_E \langle y, y \rangle^{\frac{1}{2}} x$ and let $q = {}_E \langle z, z \rangle \in E$. Then q is a projection in E . Let π be a map from B to E defined by $\pi(b) = {}_E \langle zb, z \rangle$ for any $b \in B$. Then π is an isomorphism of B onto qEq . We suppose that ρ has the Rohlin property. Then by [11, Corollary 6.4] there is a projection $p \in A_\infty$ such that $e \cdot_\rho p = \frac{1}{N}$. We regard $(X^\infty)^n$ as an $E^\infty - B^\infty$ -equivalence bimodule in the usual way. Since $p \otimes I_n \in E^\infty$, there are elements

$$u_1, \dots, u_m, v_1, \dots, v_m \in (X^\infty)^n$$

such that $p \otimes I_n = \sum_{k=1}^m {}_{E^\infty} \langle u_k, v_k \rangle$. We write that

$$u_k = (u_{k1}, \dots, u_{kn}), \quad v_k = (v_{k1}, \dots, v_{kn}),$$

where $u_{ki}, v_{ki} \in X^\infty$ for $k = 1, 2, \dots, m, i = 1, 2, \dots, n$. Thus

$$p \otimes I_n = \sum_{k=1}^m [{}_{A^\infty} \langle u_{ki}, v_{kj} \rangle]_{i,j=1}^n.$$

Hence

$$(\ast \ast \ast) \quad \sum_{k=1}^m {}_{A^\infty} \langle u_{ki}, v_{kj} \rangle = \begin{cases} p & i = j \\ 0 & i \neq j \end{cases}.$$

We note that since $p \in A_\infty$, $q(p \otimes I_n)q = q(p \otimes I_n) \in (qM_n(A)q)^\infty \cap (qM_n(A)q)'$. Let π^∞ be the isomorphism of B^∞ onto $(qM_n(A)q)^\infty$ induced by π . Let $p_1 = (\pi^\infty)^{-1}(q(p \otimes I_n)q)$. Then p_1 is a projection in B_∞ since $\pi(B) = qM_n(A)q$. We show that $e \cdot_{\sigma^\infty} p_1 = \frac{1}{N}$. Since $q = {}_E \langle z, z \rangle$,

$$\begin{aligned} q(p \otimes I_n)q &= \sum_{k=1}^m {}_{E^\infty} \langle {}_E \langle z, z \rangle u_k, {}_E \langle z, z \rangle v_k \rangle \\ &= \sum_{k=1}^m {}_{E^\infty} \langle z \langle z, u_k \rangle_{B^\infty} \langle v_k, z \rangle_{B^\infty}, z \rangle \\ &= \pi^\infty \left(\sum_{k=1}^m \langle z, u_k \rangle_{B^\infty} \langle v_k, z \rangle_{B^\infty} \right). \end{aligned}$$

Thus

$$p_1 = \sum_{k=1}^m \langle z, u_k \rangle_{B^\infty} \langle v_k, z \rangle_{B^\infty} = \sum_{k=1}^m \langle z, {}_{E^\infty} \langle u_k, v_k \rangle z \rangle_{B^\infty}.$$

Since $z \in X^n$, we write $z = (z_i)_{i=1}^n$, where $z_i \in X$ for $i = 1, 2, \dots, n$. Hence by Equation $(***)$,

$$\begin{aligned} p_1 &= \left\langle \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \sum_{k=1}^m [A^\infty \langle u_{ki}, v_{kj} \rangle]_{i,j=1}^n \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \right\rangle_{B^\infty} \\ &= \sum_{i,j=1}^n \langle z_i, \sum_{k=1}^m A^\infty \langle u_{ki}, v_{kj} \rangle z_j \rangle_{B^\infty} \\ &= \sum_{i=1}^n \langle z_i, pz_i \rangle_{B^\infty}. \end{aligned}$$

For any $w \in X$,

$$\begin{aligned} w[e \cdot_{\sigma^\infty} p_1] &= \sum_{i=1}^n w \langle [S(e_{(1)}^*) \cdot_\lambda z_i], [e_{(2)} \cdot_{\lambda^\infty} pz_i] \rangle_{B^\infty} \\ &= \sum_{i=1}^n A \langle w, [S(e_{(1)}^*) \cdot_\lambda z_i] \rangle [e_{(2)} \cdot_{\lambda^\infty} pz_i] \\ &= \sum_{i=1}^n A \langle [e_{(2)} S(e_{(1)}) \cdot_\lambda w], [S(e_{(3)}^*) \cdot_\lambda z_i] \rangle [e_{(4)} \cdot_{\lambda^\infty} pz_i] \\ &= \sum_{i=1}^n [e_{(2)} \cdot_\rho A \langle [S(e_{(1)}) \cdot_\lambda w], z_i \rangle] [e_{(3)} \cdot_{\lambda^\infty} pz_i] \\ &= \sum_{i=1}^n [e_{(2)} \cdot_{\lambda^\infty} p[S(e_{(1)}) \cdot_\lambda w] \langle z_i, z_i \rangle_B] \\ &= [e_{(2)} \cdot_{\rho^\infty} p][e_{(3)} S(e_{(1)}) \cdot_\lambda w]. \end{aligned}$$

Since $e = \sum_{i,k} \frac{d_k}{N} w_{ii}^k$,

$$\begin{aligned} w[e \cdot_{\sigma^\infty} p_1] &= \sum_{i,j,k,j_1} \frac{d_k}{N} [w_{jj_1}^k \cdot_{\rho^\infty} p][w_{j_1 i}^k S(w_{ij}^k) \cdot_\lambda w] \\ &= \sum_{j,k} \frac{d_k}{N} [w_{jj}^k \cdot_{\rho^\infty} p]w = [e \cdot_{\rho^\infty} p]w = \frac{1}{N}w. \end{aligned}$$

Thus $e \cdot_{\sigma^\infty} p_1 = \frac{1}{N}$ by Lemma 6.1. Therefore we obtain the conclusion by [11, Corollary 6.4.]. \square

Corollary 6.4. *Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B , respectively. We suppose that they are strongly Morita equivalent. Then the following hold:*

- (1) *The twisted coaction (ρ, u) has the Rohlin property if and only if so does (σ, v) ,*
- (2) *The twisted coaction (ρ, u) is approximately representable if and only if so is (σ, v) .*

Proof. (1) We suppose that (ρ, u) has the Rohlin property. Then $\widehat{\widehat{\rho}}$ has the Rohlin property by [11, Proposition 5.5]. Also, since (ρ, u) and (σ, v) are strongly Morita equivalent, $\widehat{\widehat{\rho}}$ and $\widehat{\widehat{\sigma}}$ are strongly Morita equivalent by Corollary 4.8. Thus (σ, v) has the Rohlin property by Theorem 6.3 and [11, Proposition 5.5].

(2) We suppose that (ρ, u) is approximately representable. Then $\widehat{\rho}$ has the Rohlin property by the definition of the Rohlin property and [11, Proposition 4.6]. Since (ρ, u) and (σ, v) are strongly Morita equivalent, $\widehat{\rho}$ and $\widehat{\sigma}$ are strongly Morita equivalent by Corollary 4.8. Thus by Theorem 6.3, $\widehat{\sigma}$ has the Rohlin property. Hence by the definition of the Rohlin property and [11, Proposition 4.6], (σ, v) is approximately representable. \square

7. APPLICATION

Let A and B be unital C^* -algebras and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . We suppose that A is strongly Morita equivalent to B . Let ρ be a coaction of H^0 on A . By [16, Proposition 2.1], there are an $n \in \mathbf{N}$ and a full projection $q \in M_n(A)$ such that B is isomorphic to $qM_n(A)q$. We identify B with $qM_n(A)q$. We suppose that $(\rho \otimes \text{id})(q) \sim q \otimes 1^0$ in $M_n(A) \otimes H^0$. Hence there is a partial isometry $w \in M_n(A) \otimes H^0$ such that $w^*w = (\rho \otimes \text{id})(q)$, $ww^* = q \otimes 1^0$.

Lemma 7.1. *With the above notations, there is a partial isometry $z \in M_n(A) \otimes H^0$ such that $z^*z = (\rho \otimes \text{id})(q)$, $zz^* = q \otimes 1^0$ and that $\widehat{z}(1) = q$.*

Proof. We note that $\widehat{w^*}(1) = \widehat{w}(1)^*$. Since $w^*w = (\rho \otimes \text{id})(q)$ and $ww^* = q \otimes 1^0$,

$$\widehat{w^*}(1)\widehat{w}(1) = (\text{id} \otimes \epsilon^0)((\rho \otimes \text{id})(q)) = q, \quad \widehat{w}(1)\widehat{w^*}(1) = q.$$

Let $z = (\widehat{w^*}(1) \otimes 1^0)w$. Then $\widehat{z}(1) = \widehat{w^*}(1)\widehat{w}(1) = q$. Also,

$$\begin{aligned} z^*z &= w^*(\widehat{w}(1) \otimes 1^0)(\widehat{w^*}(1) \otimes 1^0)w = (\rho \otimes \text{id})(q), \\ zz^* &= (\widehat{w^*}(1) \otimes 1^0)ww^*(\widehat{w}(1) \otimes 1^0) = q \otimes 1^0. \end{aligned}$$

Therefore, we obtain the conclusion. \square

Let

$$\begin{aligned} \sigma &= \text{Ad}(z) \circ (\rho \otimes \text{id}_{M_n(\mathbf{C})}), \\ u &= (z \otimes 1^0)(\rho \otimes \text{id}_{M_n(\mathbf{C})} \otimes \text{id}_{H^0})(z)(\text{id}_{M_n(A)} \otimes \Delta^0)(z^*). \end{aligned}$$

We note that $u \in B \otimes H^0 \otimes H^0$. We shall show that (σ, u) is a twisted coaction of H^0 on B , which is strongly Morita equivalent to ρ . We sometimes identify $A \otimes H^0 \otimes M_n(\mathbf{C})$ with $A \otimes M_n(\mathbf{C}) \otimes H^0$.

Lemma 7.2. *With the above notations, σ is a weak coaction of H^0 on B .*

Proof. For any $x \in M_n(A)$,

$$\sigma(qxq) = z(\rho \otimes \text{id})(qxq)z^* = (q \otimes 1^0)z(\rho \otimes \text{id})(x)z^*(q \otimes 1^0).$$

Hence σ is a map from B to $B \otimes H^0$. Also, by routine computations, we can see that σ is a homomorphism of B to $B \otimes H^0$ with $\sigma(q) = q \otimes 1^0$. Furthermore, since $\widehat{z}(1) = q$, for any $x \in M_n(A)$,

$$\begin{aligned} (\text{id} \otimes \epsilon^0)(\sigma(qxq)) &= (\text{id} \otimes \epsilon^0)((q \otimes 1^0)z(\rho \otimes \text{id})(x)z^*(q \otimes 1^0)) \\ &= q\widehat{z}(1)(\text{id} \otimes \epsilon^0)((\rho \otimes \text{id})(x))\widehat{z}^*(1)q = qxq. \end{aligned}$$

Thus σ is a weak coaction of H^0 on B . \square

Lemma 7.3. *With the above notations, (σ, u) is a twisted coaction of H^0 on B .*

Proof. By routine computations, we can see that $uu^* = u^*u = q \otimes 1^0 \otimes 1^0$. Thus u is a unitary element in $B \otimes H^0 \otimes H^0$. For any $x \in M_n(A)$,

$$\begin{aligned} ((\sigma \otimes \text{id}_{H^0}) \circ \sigma)(qxq) &= (z \otimes 1^0)(\rho \otimes \text{id} \otimes \text{id}_{H^0})(z)((\rho \otimes \text{id} \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}))(qxq) \\ &\times (\rho \otimes \text{id} \otimes \text{id}_{H^0})(z^*)(z^* \otimes 1^0). \end{aligned}$$

On the other hand,

$$\begin{aligned} (\text{Ad}(u) \circ (\text{id} \otimes \Delta^0) \circ \sigma)(qxq) &= (z \otimes 1^0)(\rho \otimes \text{id} \otimes \text{id}_{H^0})(z)((\text{id} \otimes \Delta^0) \circ (\rho \otimes \text{id}))(qxq) \\ &\times (\rho \otimes \text{id} \otimes \text{id}_{H^0})(z^*)(z^* \otimes 1^0). \end{aligned}$$

Since $(\rho \otimes \text{id} \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}) = (\text{id} \otimes \Delta^0) \circ (\rho \otimes \text{id})$, we obtain that

$$(\sigma \otimes \text{id}_{H^0}) \circ \sigma = \text{Ad}(u) \circ (\text{id} \otimes \Delta^0) \circ \sigma.$$

Also,

$$\begin{aligned} (u \otimes 1^0)(\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0})(u) &= (z \otimes 1^0 \otimes 1^0)(\rho \otimes \text{id} \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(z \otimes 1^0) \\ &\times (\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0})((\rho \otimes \text{id} \otimes \text{id}_{H^0})(z)(\text{id} \otimes \Delta^0)(z^*)). \end{aligned}$$

On the other hand, since $(\rho \otimes \text{id} \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}) = (\text{id} \otimes \Delta^0) \circ (\rho \otimes \text{id})$,

$$\begin{aligned}
& (\sigma \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u)(\text{id} \otimes \text{id}_{H^0} \otimes \Delta^0)(u) \\
&= (z \otimes 1^0 \otimes 1^0)(\rho \otimes \text{id} \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(z \otimes 1^0) \\
&\times (\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0})((\rho \otimes \text{id} \otimes \text{id}_{H^0})(z)) \\
&\times (\rho \otimes \text{id} \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})((\text{id} \otimes \Delta^0)(z^*)) \\
&\times (\text{id} \otimes \text{id}_{H^0} \otimes \Delta^0)((\rho \otimes \text{id} \otimes \text{id}_{H^0})(z))(\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0})((\text{id} \otimes \Delta^0)(z^*)).
\end{aligned}$$

We can see that

$$(\rho \otimes \text{id} \otimes \text{id}_{H^0} \otimes \text{id}_{H^0}) \circ (\text{id} \otimes \Delta^0) = (\text{id} \otimes \text{id}_{H^0} \otimes \Delta^0) \circ (\rho \otimes \text{id} \otimes \text{id}_{H^0})$$

by easy computations. Furthermore, we note that

$$\begin{aligned}
& (\text{id} \otimes \text{id}_{H^0} \otimes \Delta^0) \circ (\text{id} \otimes \Delta^0) \circ (\rho \otimes \text{id}) \\
&= (\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0}) \circ (\text{id} \otimes \Delta^0) \circ (\rho \otimes \text{id}) \\
&= (\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id} \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}).
\end{aligned}$$

Thus since

$$\begin{aligned}
& (\text{id} \otimes \text{id}_{H^0} \otimes \Delta^0)((\rho \otimes \text{id} \otimes \text{id}_{H^0})((\rho \otimes \text{id})(q))) \\
&= (\text{id} \otimes \text{id}_{H^0} \otimes \Delta^0)((\text{id} \otimes \Delta^0)((\rho \otimes \text{id})(q))), \\
& (\sigma \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u)(\text{id} \otimes \text{id}_{H^0} \otimes \Delta^0)(u) \\
&= (z \otimes 1^0 \otimes 1^0)(\rho \otimes \text{id} \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(z \otimes 1^0) \\
&\times (\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0})((\rho \otimes \text{id} \otimes \text{id}_{H^0})(z)) \\
&\times (\text{id} \otimes \text{id}_{H^0} \otimes \Delta^0)((\rho \otimes \text{id} \otimes \text{id}_{H^0})((\rho \otimes \text{id})(q))) \\
&\times (\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0})((\text{id} \otimes \Delta^0)(z^*)) \\
&= (z \otimes 1^0 \otimes 1^0)(\rho \otimes \text{id} \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(z \otimes 1^0) \\
&\times (\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0})((\rho \otimes \text{id} \otimes \text{id}_{H^0})(z)(\text{id} \otimes \Delta^0)(z^*)).
\end{aligned}$$

Hence we obtain that

$$(u \otimes 1^0)(\text{id} \otimes \Delta^0 \otimes \text{id}_{H^0})(u) = (\sigma \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u)(\text{id} \otimes \text{id}_{H^0} \otimes \Delta^0)(u).$$

Furthermore, since $\widehat{z}(1) = q$, for any $h \in H$,

$$\begin{aligned}
& (\text{id} \otimes h \otimes \epsilon^0)(u) = \widehat{z}(h_{(1)})[h_{(2)} \cdot_{\rho \otimes \text{id}} q] \widehat{z}^*(h_{(3)}) = (\text{id} \otimes h)(\sigma(q)) = \epsilon(h)q, \\
& (\text{id} \otimes \epsilon^0 \otimes h)(u) = \widehat{z}(1)[1 \cdot_{\rho \otimes \text{id}} \widehat{z}(h_{(1)})] \widehat{z}^*(h_{(2)}) = \widehat{z}(1)\epsilon(h) = \epsilon(h)q.
\end{aligned}$$

Therefore, (σ, u) is a twisted coaction of H^0 on B . \square

Let f be a minimal projection in $M_n(\mathbf{C})$ and let p be a full projection in $M_n(A)$ defined by $p = 1_A \otimes f$. Let $X = pM_n(A)q$. We regard X as an $A - B$ -equivalence bimodule in the usual way, where we identify A and

B with $pM_n(A)p$ and $qM_n(A)q$, respectively. Then we can regard X as a set $\{[a_1, \dots, a_n]q \mid a_i \in A, i = 1, \dots, n\}$. Let λ be a linear map from X to $X \otimes H^0$ defined by

$$\begin{aligned}\lambda([a_1, \dots, a_n]q) &= [\rho(a_1), \dots, \rho(a_n)](\rho \otimes \text{id})(q)z^* \\ &= [\rho(a_1), \dots, \rho(a_n)]z^*(q \otimes 1^0)\end{aligned}$$

for any $[a_1, \dots, a_n]q \in X$.

Lemma 7.4. *With the above notations, λ is a twisted coaction of H^0 on X with respect to (A, B, ρ, σ, u) .*

Proof. By routine computations, we can see that λ is a weak coaction of H^0 on X with respect to (A, B, ρ, σ, u) . For any $[a_1, \dots, a_n]q \in X$,

$$\begin{aligned}((\lambda \otimes \text{id}_{H^0}) \circ \lambda)([a_1, \dots, a_n]q) &= [((\rho \otimes \text{id}_{H^0}) \circ \rho)(a_1), \dots, ((\rho \otimes \text{id}_{H^0}) \circ \rho)(a_n)] \\ &\quad \times (\rho \otimes \text{id} \otimes \text{id}_{H^0})(z^*)(z^* \otimes 1^0).\end{aligned}$$

On the other hand, since $(\rho \otimes \text{id}_{H^0}) \circ \rho = (\text{id} \otimes \Delta^0) \circ \rho$,

$$\begin{aligned}((\text{id} \otimes \Delta^0) \circ \lambda)([a_1, \dots, a_n]q)u^* &= [((\text{id} \otimes \Delta^0) \circ \rho)(a_1), \dots, ((\text{id} \otimes \Delta^0) \circ \rho)(a_n)] \\ &\quad \times (\rho \otimes \text{id} \otimes \text{id}_{H^0})(z^*)(z^* \otimes 1^0).\end{aligned}$$

Hence for any $[a_1, \dots, a_n]q \in X$,

$$((\lambda \otimes \text{id}_{H^0}) \circ \lambda)([a_1, \dots, a_n]q) = ((\text{id} \otimes \Delta^0) \circ \lambda)([a_1, \dots, a_n]q)u^*.$$

Thus λ is a twisted coaction of H^0 on X with respect to (A, B, ρ, σ, u) . \square

Theorem 7.5. *Let A be a unital C^* -algebra and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let ρ be a coaction of H^0 on A with the Rohlin property. Let q be a full projection in a C^* -algebra $M_n(A)$ such that*

$$(\rho \otimes \text{id}_{M_n(\mathbb{C})})(q) \sim q \otimes 1^0$$

in $M_n(A) \otimes H^0$. Let $B = qM_n(A)q$. Then there is a coaction of H^0 on B with the Rohlin property.

Proof. By Lemmas 7.3 and 7.4, there is a twisted coaction (σ, u) such that (σ, u) is strongly Morita equivalent to ρ . By Corollary 6.4, (σ, u) has the Rohlin property. Furthermore, by [11, Theorem 9.6], there is a unitary element $y \in B \otimes H^0$ such that

$$(y \otimes 1^0)(\sigma \otimes \text{id}_{H^0})u(\text{id} \otimes \Delta^0)(y^*) = 1_B \otimes 1^0 \otimes 1^0.$$

Let $\sigma_1 = \text{Ad}(y) \circ \sigma$. Then σ_1 is a coaction of H^0 on B with the Rohlin property by easy computations since σ_1 is exterior equivalent to (σ, u) . \square

Let A be a UHF-algebra of type N^∞ , where N is the dimension of a finite dimensional C^* -Hopf algebra H . In [11], we showed that there is a coaction ρ of H^0 on A with the Rohlin property.

Corollary 7.6. *With the above notations, for any unital C^* -algebra B , that is strongly Morita equivalent to A , there is a coaction σ of H^0 on B with the Rohlin property.*

Proof. By [16, Proposition 2.1] there are $n \in \mathbf{N}$ and a full projection $q \in M_n(A)$ such that B is isomorphic to $qM_n(A)q$. We identify B with $qM_n(A)q$. Let ρ be a coaction of H^0 on A with the Rohlin property. Then [11, Lemma 10.10], $(\rho \otimes \text{id}_{M_n(\mathbf{C})})(q) \sim q \otimes 1^0$ in $M_n(A) \otimes H^0$ since A has cancellation. Therefore, by Theorem 7.5 we obtain the conclusion. \square

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REFERENCES

- [1] B. Blackadar, *K-theory for operator algebras*, M. S. R. I. Publications 5, 2nd Edition, Cambridge Univ. Press, Cambridge, 1998.
- [2] R. J. Blattner, M. Cohen and S. Montgomery, *Crossed products and inner actions of Hopf algebras*, Trans. Amer. Math. Soc., **298** (1986), 671–711.
- [3] F. Combes, *Crossed products and Morita equivalence*, Proc. London Math. Soc., **49** (1984), 289–306.
- [4] R. E. Curto, P. S. Muhly and D. P. Williams, *Cross products of strong Morita equivalent C^* -algebras*, Proc. Amer. Math. Soc., **90** (1984), 528–530.
- [5] M. Z. Guo and X. X. Zhang, *Takesaki-Takai duality theorem in Hilbert modules*, Acta Mth. Sin., Engl. Ser. **20** (2004) 1079–1088.
- [6] M. Izumi, *Finite group actions on C^* -algebras with the Rohlin property-I*, Duke Math. J., **122** (2004) 233–280.
- [7] S. Jansen and S. Waldmann, *The H -covariant strong Picard groupoid*, J. Pure Appl. Algebra, **205** (2006), 542–598.
- [8] T. Kajiwara and Y. Watatani, *Jones index theory by Hilbert C^* -bimodules and K -Theory*, Trans. Amer. Math. Soc., **352** (2000), 3429–3472.

- [9] T. Kajiwara and Y. Watatani, *Crossed products of Hilbert C^* -bimodules by countable discrete groups*, Proc. Amer. Math. Soc., **126** (1998), 841–851.
- [10] K. Kodaka and T. Teruya, *Inclusions of unital C^* -algebras of index-finite type with depth 2 induced by saturated actions of finite dimensional C^* -Hopf algebras*, Math. Scand., **104** (2009), 221–248.
- [11] K. Kodaka and T. Teruya, *The Rohlin property for coactions of finite dimensional C^* -Hopf algebras on unital C^* -algebras*, J. Operator Theory, **74** (2015), 101–142, to appear.
- [12] E. C. Lance, *Hilbert C^* -modules*, A toolkit for operator algebraists, London Math. Soc. Lecture Note Series, **210**, Cambridge Univ. Press, Cambridge, 1995.
- [13] H. Osaka, K. Kodaka and T. Teruya, *The Rohlin property for inclusions of C^* -algebras with a finite Watatani index*, Operator structures and dynamical systems, 177–185, Contemp. Math., **503** Amer. Math. Soc., Providence, RI, 2009.
- [14] J. A. Packer, *C^* -algebras generated by projective representations of the discrete Heisenberg group*, J. Operator Theory, **18** (1987), 41–66.
- [15] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^* -algebras*, Mathematical Surveys and Monographs, **60**, Amer. Math. Soc., 1998.
- [16] M. A. Rieffel, *C^* -algebras associated with irrational rotations*, Pacific J. Math., **93** (1981), 415–429.
- [17] W. Szymański and C. Peligrad, *Saturated actions of finite dimensional Hopf $*$ -algebras on C^* -algebras*, Math. Scand., **75** (1994), 217–239.
- [18] Y. Watatani, *Index for C^* -subalgebras*, Mem. Amer. Math. Soc., **424** (1990).

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